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MATHEUS JANCZKOWSKI FOGAÇA

A SYSTEMATIC APPROACH TO SOLVE ORDINARY DIFFERENTIAL EQUATIONS USING A GENERALIZATION OF THE INTEGRATING FACTOR

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Dissertação apresentada ao Programa de Pós-Graduação em Engenharia Mecânica do Centro de Ciências Tecnológicas da Universidade do Estado de Santa Catarina, como parte dos requisitos necessários para a obtenção do título de Mestre em Engenharia Mecânica.

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ABSTRACT

Linear ordinary differential equations (ODEs) appear in many real-world problems, such as in Engineering and in Applied Mathematics. ODEs and systems of coupled ODEs are usually difficult and cumbersome to solve, specially if the coefficients are function of the independent variable. For this reason, even when the coefficients are constant, numerical methods are usually chosen, despite their computational cost and sources of error. Analytical approaches - like variation of parameters, undetermined coefficients, Laplace and Fourier transforms - either need a candidate solution or need complicated algebraic and inverse operations.

To tackle this problem, the Generalized Integrating Factor (GIF) is proposed, which is a generalization of the Leibniz integrating factor. The new method can solve linear ODEs of n-th order by a reduction order approach. The method is presented for ODE of general order, although this work focuses on second order ODEs, with homogeneous and the particular solutions written as the product of convolutions. Then, the technique is particularized for the case of constant coefficients and for a set of important continuous and discontinuous excitation functions. The advantages of exactness and low computational cost are presented with numerical experiments and complexity analysis of the associated algorithms.

As the particular solution depends on the analytical solution of a convolution, it must be particularized for some sets of excitations. Also, not all excitations are easy (or even possible) to convolute analytically. Thus, Heaviside Series (HS) method is proposed as an extension to the Generalized Integrating Factor. In this new approach, the excitation function is approximated using a finite series of Heaviside steps multiplied by polynomial terms. The proposed approach can be seen as a mixed approach, where the homogeneous solution is analytical and the particular solution is the analytical solution to an approximate excitation. Using numerical experiments, a high rate of convergence (between 2 and 4) is shown, as well as low computational cost. Thus, the formulated family of methods prove to be a reliable and cheap way to solve linear ODEs in real-world problems.

Throughout the derivation of the GIF and the HS methods for systems of coupled linear ODEs with constant matrix coefficients, the hypothesis of classical normal modes was used to introduce some simplifications to the solution procedure. It is shown, nonetheless, that this hypothesis is not necessary for the proposed family of methods to work and, the particularization of the presented solutions to non-classical normal modes is derived. Most importantly, this work shows that the computational cost does not increase much when the modes are not classical normal, thus, enabling the use of the techniques even in this situation.

For this reason, the solutions for the homogeneous solution and the particular solutions due to

Dirac's delta and due to HS are explicitly provided, so more complicated excitation functions can be constructed using them. It is proven how the expressions naturally become the formulae previously derived for classical normal modes, thus, showing the consistency of the approach. Besides, algorithms are provided to indicate the efficient computer implementation of the methods.

Keywords: Differential equations; Systems of differential equations; Analytical method; Semi-analytical method; Generalized integrating factor; Heaviside Series; Non-classical normal modes.

RESUMO

Equações diferenciais ordinárias (EDOs) lineares aparecem em muitos problemas reais, como em Engenharia e e em Matemática Aplicada. EDOs e sistems de EDOs acopladas são comumente difíceis e complicadas de se resolver, especialmente quando os coeficientes são funções da variável independente. Em virtude disso, mesmo quando os coeficientes são constantes, métodos numéricos são frequentemente escolhidos, apesar do custo computacional e das fontes de erro. Abordagens analíticas - como variação de parâmetros, coeficientes indeterminados, transformadas de Laplace e de Fourier - necessitam de uma solução candidato ou necessitam de complicadas operações algébricas e de inversão.

Para atacar este problema, o Fator Integrante Generalizado (GIF, do inglês *Generalized Integrating Factor*) é proposto, o qual é uma generalização do fator integrante de Leibniz. O novo método pode solucionar EDOs lineares de ordem n lançando mão de uma abordagem de redução de ordem. O método é apresentado para EDOs de ordem geral, embora este trabalho foque em EDOs de segunda ordem, em que as soluções homogênea e particular são o produto de convoluções. Em seguida, a técnica é particularizada para o caso de coeficientes constantes e para um conjunto de importantes funções de excitação, tanto contínuas como descontínuas. As vantagens de exatidão e baixo custo computacional são apresentadas com experimentos numéricos e com a análise de complexidade dos algoritmos associados.

Como a solução particular depende da solução analítica de uma convolução, esta deve ser particularizada para alguns conjuntos de excitação. Mesmo assim, nem todas as excitações têm a convolução facilmente (sequer é possível) solucionada analiticamente. Logo, o método Séries de Heaviside (HS, do inglês *Heaviside Series*) é proposto como uma extensão ao GIF. Nesta nova abordagem, a função de excitação é aproximada usando uma série finita de degraus de Heaviside multiplicados por funções polinomiais. A abordagem proposta pode ser visualizada como uma abordagem mista, já que a solução homogênea é analítica e a solução particular é analítica sobre uma excitação aproximada. Utilizando experimentos numéricos, altas taxas de convergência foram medidas (entre 2 e 4), assim como um baixo custo computacional. Portanto, a família de métodos proposta prova a si mesma como confiável e barata na atividade de resolver EDOs lineares em problemas reais.

No decorrer da dedução dos métodos GIF e HS para a solução de sistemas de EDOs lineares acopladas com coeficientes matriciais constantes, a hipótese de modos normais clássicos foi utilizada para introduzir algumas simplificações no processo de solução. Esta hipótese, entretanto, não é necessária para fazer a família de métodos proposta funcionar, e, em consequência, uma particularização para a solução quando os modos de de vibrar não são normais clássicos é apresentada. De forma ainda mais importante, este trabalho mostra que o custo computacional não aumenta muito quando os modos não são normais clássicos, logo, a aplicação destes métodos continua interessante mesmo nesta situação.

Por esta razão, as soluções homogênea e particular, no caso de Delta de Dirac e do HS, são explicitamente fornecidas, tal que funções de excitação mais complexas possam ser construídas pelo uso destas. É provado também que as expressões naturalmente retornam às expressões anteriores quando os modos de vibrar são normais clássicos, o que mostra a consistência da abordagem. Além disso, algoritmos são apresentados para indicar uma forma eficiente de implementar os métodos.

Palavras-chave: Equações diferenciais; Sistemas de equações diferenciais; Método analítico; Método semi-analítico; Fator integrante generalizado; Séries de Heaviside; Modos de vibrar não normais clássicos.

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LIST OF ABBREVIATIONS AND ACRONYMS

ODE	Ordinary Differential Equation
DOF	Degree Of Freedom
GIF	Generalized Integrating Factor
HS	Heaviside Series
FEM	Finite Element Method
FEA	Finite Element Analysis
BEM	Boundary Element Method
BEA	Boundary Element Analysis
RLC	Resistor-Inductor-Capacitor (Circuit)
RHS	Right Hand Side
LHS	Left Hand Side

CONTENTS

1 INTRODUCTION

Linear and non-homogeneous ordinary differential equations (ODE) are pervasive in many areas of applied mathematics. In particular, second-order ordinary differential equations arise in a variety of practical problems in mathematics, physics, and engineering, such as the dynamic equilibrium of mechanical systems and viscoelasticity modeling. While the solution for constant coefficients and periodic forces is well known (RAO, 2017), finding the solution for general problems can be challenging.

Solving second-order ordinary differential equations is a well-established topic with a rich history. Numerous techniques and methods have been developed over the years to solve second-order ODEs, including analytical, numerical, and approximate methods. Each method presents an intrinsic disadvantage, which can be the need to find particular or candidate solutions; cumbersome inverse and algebraic operations; increase in dimensionality of the problem and many more. These disadvantages make the solution and analysis of second order ODEs costly and not straightforward.

Analytical methods involve finding an exact solution to the ODE in terms of elementary functions such as polynomials, trigonometric functions, and exponential functions. Analytical solutions are particularly useful when the ODE has a simple form or when we are interested in understanding the behavior of the solution explicitly. An example is the use of computational techniques to solve the ODE symbolically, such as the symbolic integration algorithm implemented in the Mathematica software (RESEARCH, 2018).

Numerical methods involve approximating the solution to the ODE using numerical algorithms. An example is the use of high-order methods that achieve high accuracy with fewer computational resources, such as the spectral collocation method and the spectral deferred correction method (BOYD, 2001; KARNIADAKIS; SHERWIN, 2013). A very recent approach is the use of machine learning techniques, such as the neural network method, which can learn the solution to the ODE directly from data (RAISSI; KARNIADAKIS, 2018; BONNAFFé; COULSON, 2023).

Approximate methods involve finding an approximate solution to the ODE that satisfies certain criteria. Examples include the use of asymptotic methods (BENDER; ORSZAG, 1999) and numerical techniques to construct approximate solutions, such as the generalized polynomial chaos method, which constructs an approximate solution as a series of orthogonal polynomials (XIU; KARNIADAKIS, 2002).

Stemming from the vibration and electric circuitry analysis and from control theory, coupled systems of second order ODEs are quite important (ABELL; BRASELTON, 2023; KREYSZIG; KREYSZIG; NORMINTON, 2011; BOYCE; DIPRIMA, 2001; ROWELL, a; ROWELL, b). These systems behave much like ODEs, but the solution lies in a vector space of dimension n, the number of degrees of freedom (DOF) of the system.

In the particular case of mechanical vibrations, the solution is given by the following system of ODEs,

$$\mathbf{M}\ddot{\mathbf{y}}(t)+\mathbf{C}\dot{\mathbf{y}}(t)+\mathbf{K}\mathbf{y}(t)=\mathbf{f}(t), \tag{1}$$

where $\mathbf{y}(t)$ is the vector of dependent variables, t is the independent variable, **M**, **C** and **K** are coefficient matrices, which can be calculated using direct analysis over a discrete mechanical system or using Finite Element Analysis (FEA). In other engineering applications, such as electric circuitry, these matrices can also be derived in similar fashion, although they mean different physical properties of the phenomenon.

In a broad sense, when the matrix coefficients are constant, under some circumstances regarding **C**, it may be possible to convert the original coupled problem into n uncoupled problems (KAWANO et al., 2018). When coefficient matrix **C** is written as a linear combination of **M** and **K** one can use a change of basis given by the eigenvectors of the generalized eigenvalue problem (modal problem), resulting in n independent one-dimensional second order ODEs (RAO, 2017). Extensions to other general viscous coefficient matrix **C** involve the solution of a quadratic eigenvalue problem and are discussed by (MA; MORZFELD; IMAM, 2010). This general procedure is known as Modal Superposition and it is common to consider just a subset of the n eigenvectors to exclude (usually larger) modes in the response and to avoid the expensive computation of all the eigenpairs. Nonetheless, each individual equation still needs to be solved either analytically or numerically.

Another approach to solve these systems of ODEs analytically is to re-write Eq. (1) by using state variables, converting a coupled second order equation to an extended system of first order coupled ODEs (CHAHANDE; ARORA, 1994). The main drawback is that the new first order system of coupled ODEs has twice the dimensionality. Analytical solutions can be derived by using exponential maps (LOBO; JOHNSON, 2013).

The most common approach to solve the problem defined in Eq. (1) is by using some sort of numerical procedure, like Central Difference, Newmark-Beta, Wilson-θ, among other methods (HUGHES, 2000). In general terms, all these methods rely on some type of hypothesis on the behaviour of the response and its derivatives between two or more discrete times, t_m and t_{m+1}, also known as temporal discretization. The set of hypotheses defines each numerical method and its solution procedure.

It is worth noticing that such hypotheses introduce approximations to the real responses. Thus, one can identify some potential error sources for any numerical discrete method. The first source is known as interpolation error, since each particular method makes assumptions about the behavior of the solution and/or its derivatives between the discrete times. The order of accuracy is dependent on the order of the approximations assumed by each method. Other source of error is due to the time discretization, since a coarse grid can hinder some important aspects of the problem. The time discretization is also an issue for conditionally stable methods, like Central Difference. Finally, as the equations are solved by means of some numerical procedure, there is the possibility of numerical errors.

To better predict these behaviors in numerical methods, their update rule can be summarized as

$$\mathbf{s}_{t+1} = \mathbf{A}\mathbf{s}_t + \mathbf{L}_t, \tag{2}$$

where $\mathbf{s}$ can contain the response $\mathbf{y}$ and its derivatives, $\mathbf{A}$ is an amplification matrix and $\mathbf{L}_t$ is associated to loading. Amplification matrix $\mathbf{A}$ is a discrete operator depending on matrices $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$, the step size Δt and the approximations used to construct each method. All the properties like stability, artificial damping, artificial modification of frequencies, to name a few, can be studied from $\mathbf{A}$.

Regarding stability, a method can be *unconditionally stable* when the response $\mathbf{s}_{t+1}$ is bounded for any step size Δt; otherwise, it is *conditionally stable*. An example of the later is discussed by (MIMOUNA; TCHELEPI, 2019), while there has been a great effort in this research field to devise new unconditionally stable methods, such as (NEWMARK, 1959; HOUBOLT, 1950; HILBER; HUGHES; TAYLOR, 1977; WILSON; FARHOOMAND; BATHE, 1972; WOOD; BOSSAK; ZIENKIEWICZ, 1980; KATONA; ZIENKIEWICZ, 1985; ZIENKIEWICZ et al., 1984; FAN; FUNG; SHENG, 1997).

Targeting higher order interpolations and higher accuracy with less computational cost, many new methods have been proposed in the last two decades. Some examples for explicit methods are (KIM; REDDY, 2020; LIU et al., 2021; MALAKIYEH et al., 2023) and for implicit methods (LI; YU, 2019; LI; LI; YU, 2020; NOH; BATHE, 2018; WEN et al., 2017; MALAKIYEH; SHOJAEE; BATHE, 2019; BATHE, 2007; SOARES, 2015; SHOJAEE; ROSTAMI; ABBASI, 2015).

A rather new approach is to use exponential maps to solve higher order ODEs, since exponential of matrix form solutions to first order systems of coupled ODEs (GAO; NIE, 2021; BARUCQ; DURUFLé; N'DIAYE, 2018). Some of these approaches use state variables to lower the order of the original system of coupled ODEs and, then, integrate it (SONG et al., 2023; WANG; AU, 2007). The efficient evaluation of the exponential maps rely on approximations, like Padé and Chebyshev approximations. This approach is meant to be highly accurate and does not present numerical dissipation without much more computational cost when compared to Newmark method (SONG et al., 2023).

Due to the intrinsic difficulty in solving such ODEs, that are highly applicable in Engineering problems, this work proposes a new analytical method, which is a generalization of the traditional Leibniz integrating factor, thus, it is called the Generalized Integrating Factor (GIF). The reason for extending the well-established Leibniz integrating factor is due to its analytical solutions to first order linear ODEs, regardless of the coefficients. The proposed method can be used for ODEs of general order and results in integrating factors that are functions of the independent variable only, disregarding the coefficients of the ODE. This contrasts with integrating factors that depend on both the dependent and independent variables, as in (CHEB-TERRAB; ROCHE, 1999).

The proposed integrating factor in this work is a reduction order technique. For each factor found and the subsequent integration, the order of the original differential equation is reduced by one. Therefore, the technique can be used recursively until the order is zero, and the original Leibniz integrating factor turns out to be a particular case. Hence, in contrast to the original method, that can be used only to first order differential equations, the proposed generalization can be used to higher orders. Thereby, the main novelty of this work is to propose a generalization that can systematically find analytical complete solutions of linear ODEs with or without constant coefficients.

The GIF method can be applied to solve both homogeneous and non homogeneous differential equations without requiring the knowledge of the homogeneous solution or the need to propose a candidate solution, as is the case with the variation of parameters and the method of undetermined coefficients, respectively (BOYCE; DIPRIMA, 2001; LEWIS; ONDER; PRUDIL, 2022). The complete solution of the differential equation can be obtained as the sum of the homogeneous and particular solutions, which are obtained simultaneously using nested convolutions. This approach can be called a *systematic* method and can be used to solve a wide range of scientific and engineering problems that would, otherwise, rely on the variation of parameters or the method of undetermined coefficients, as well as integral transforms like Laplace and Fourier (BOYCE; DIPRIMA, 2001; LEWIS; ONDER; PRUDIL, 2022; KITTIPOOM, 2019; QUINN; RAI, 2012; NAZMUL; DEVNATH, 2020; MAHMOODI; ZOLFAGHARI; MINUCHEHR, 2019; ZHAO et al., 2022; KELLY, 2006). Each one of those methods present a disadvantage, like the need for a candidate solution, the need for the homogeneous solution, complicated algebraic operations, or the need of an inverse transform.

The integrating factor itself is found at every reduction order step as a particular solution of a nonlinear differential equation, eliminating the need to find the complete solution of the nonlinear equation and to define an initial value problem. This result shows that the GIF builds a bridge between linear and nonlinear differential equations, enabling a two-sided path between both. For instance, a linear equation can be solved by finding a particular solution to a nonlinear equation and vice-versa. It is shown, however, that, for many families of ODEs, the particular solution to the sister nonlinear ODE is easilly found, as are the cases of Euler-Cauchy, Bessel and constant coefficients.

Due to already cited applications in real-world problems, second order ODEs with constant coefficients, both scalar and matrix coefficients, will be the main focus of this work after presenting the general formulation. The homogeneous and particular solutions are obtained by double convolutions, which yield a sum of exponential maps when the coefficients are constant. Exponential maps are essentially exponential of matrices, that particularize for the conventional exponential function when the matrix is 1×1, *i.e.* a scalar, (GALLIER, 2011).

Although exponential maps are expensive to evaluate (HIGHAM, 2008), a rich field of research has been developed around its numerical evaluation in the past decade, hence, it is another motivation for the analytical solution of higher order linear differential equations, due to the advance in exponential integrators and matrix exponentiation. In the past two decades, a lot of effort has been put on exponential integrators for their accuracy (GAUDREAULT; RAINWATER; TOKMAN, 2018), hence, a multitude of methods have been developed to reduce computation cost and maximize the accuracy to cost ratio, thus, enabling the widespread use of exponential integrators and analytic methods instead of explicit numerical methods. The computational cost savings are mainly in calculating matrix exponentials, such as (SIDJE, 1998; BOTCHEV; KNIZHNERMAN, 2020; GAUDREAULT; RAINWATER; TOKMAN, 2018; SASTRE; IBáñEZ; DEFEZ, 2019), and for sparse matrices as well (VO; SIDJE, 2017). This advance in matrix exponentiation makes the GIF a practical choice for solving systems of coupled differential equations exactly.

Distinct useful non-homogeneous terms are studied for the particular solutions, which have direct application in applied mathematics and engineering, such as periodic, polynomial, Dirac's delta and Heaviside excitations. For all these excitation functions, analytical particular solutions were found using the double convolutions. For the continuous excitation types, like periodic and polynomial functions, the particular solutions does not depend upon the exponential maps; while, for the discontinuous excitation functions, like Dirac's delta and Heaviside step, they do, much like the homogeneous response.

Nonetheless, when the double convolutions are not practical to evaluate for a specific excitation function or when simply there is no interest in evaluating them, the original excitation function can be constructed using the aforementioned functions. To this aid, the Heaviside step is particularly useful, since it can be multiplied by polynomials and, then, used to approximate functions locally using polynomial terms.

The newly proposed form of function approximation is called Heaviside Series (HS), since the representation is carried out using finite sums of Heaviside step functions multiplied by polynomial coefficients. This technique is worth by itself, since it can make polynomial approximations of functions that are not tied to a single approximation point and, consequently, to a single limiting radius of convergence. This technique is used with the GIF to generate unconditionally stable and accurate solutions to systems of linear ODEs. As shown in C.3 for Rayleigh proportional damping, when the solution is not stable, it means that the system is physically unstable, hence, there is no artificial numerical dissipation in the HS method and the technique does not mask ill-conditioned models.

This approach can be seen as a further generalization of the GIF method to more general forms of excitation, since it can be used when the convolution is problematic. The resulting methodology is an hybrid method for the solution is analytical, but the excitation function is approximated. For this reason, the HS method can be classified as semi-analytical.

The computational implementation of the GIF and of the HS methods was made public and is available in two GitHub repositories, each dedicated to one of the methods. GIF is available in <https://github.com/CodeLenz/Giffndof.jl>, while HS in <https://github.com/CodeLenz/HeavisideSeriesODE>. There, usage examples are provided, as well as the documentation and the source files. The Julia programming language, (BEZANSON et al., 2012), was used and it is fully open-source. Julia is a fast and optimized programming language that was developed in MIT, it was used to code and execute all the numerical experiments of this work.

The work is organized in 4 main chapters: the formulation of the GIF for linear ODEs with a single DOF, the extension to systems of coupled linear ODEs, the introduction of the HS method and the study of the GIF and the HS methods to problems with non-classical normal modes. Each chapter presents the careful mathematical derivation step by step, along with numerical examples provided to show and compare the accuracy of the techniques and the correctness of the computer implementation. As one of the appeals of the proposed family of methods is the computational efficiency, algorithmic complexity analysis are derived and numerical experimental are carried out to evaluate and compare the computational effort to well-established techniques, such as Newmark-beta and State Variables.

The results are sound and corroborate that the GIF and HS methods are indeed accurate and fast. The tests were performed both for the number of time steps and for the dimensionality of the problem. To this aid, the Finite Element Analysis (FEA) of a metallic truss is used, since the dimensionality can be raised by increasing the number of modes. It is experimentally shown that the proposed methods take lesser computational effort, even when the number of DOFs is increased. Besides, it is also shown that the HS method, for instance, has a convergence rate between 2 and 4, against the 1 and 2 of the Newmark-beta approach.

Summing up the results, one can observe that the GIF and the HS methods are reliable and efficient, which make them a suitable choice to solve practical problems involving linear ODEs in Engineering, such as vibration analysis, electric circuitry analysis and control. For future works, a set of research ideas are left, such as: extension to non linear problems, application to modal analysis, application in Optimization and application in Boundary Element Analysis (BEA), in acoustics for example.

This work is the binding of 4 papers and they shape the conceptual and textual organization. A brief description of these individual works is given in the following. Two papers have already been submitted to a journal and are in revision. These two works contemplate Chapter 2 and Chapter 3 respectively. There is a third paper that has already been written and will be submitted right after the publication of the first two. The third paper discusses solely the HS method and, thus, makes Chapter 4. Finally, Chapter 5, concerned with non-classical normal modes, will form a fourth paper to be submitted when the third one is published. Chapter 5 is the least matured work and lacks, above all, numerical experiments.

Each chapter has its own introduction to enumerate shortly what is expected of the chapter. Each chapter also has a final section of *Final remarks*, that is effectively a section of conclusions and will do a summary of the achievements of the chapter. In the end, though, a proper chapter of Conclusion is presented to conclude the work as a whole. Each chapter has an ensemble of mathematical ideas and proofs that are left in appendices. These appendices are organized in the end of the work, past the references.

Regarding the traditional List of Symbols section, there is no List of Symbols. This dissertation has more than 700 Equations and plenty of different symbols both in scalar form or in boldface. There are many different indexes and some symbols (like a) are used in different contexts. To this end, the author decided to carefully introduce each symbol and its meaning along the text, depending on the context. Special care was taken to not change symbols when addressing an Appendix.

1.1 JUSTIFICATION

Ordinary differential equations, both in a single dimension and in n-dimensional spaces, are fundamental to model a range of natural phenomena with importance in Engineering. Three main examples are dynamic response of mechanical systems, analysis of electric circuitry, and control theory. Solving these ODEs is challenging in general application, since different classes of problems require different methods, thus, there is a lack of a more systematic method to analytically solve them, especially second order ODEs.

Hence, the aim of this work is to establish a new family of methods to solve ODEs that can be systematically used to generate analytical solutions, thus improving simulation and optimization capabilities. Besides, this family of methods must be computationally efficient, hence, enabling faster and more accurate solutions. This performance allows immediate use in traditionally computational intensive applications, such as Structural Optimization and BEA, to name a few.

1.2 OBJECTIVES

1.2.1 General Objective

This dissertation proposes a new family of methods to analytically solve linear ODEs and systems of coupled linear ODEs. The application and characterization of the technique towards second order problems with constant coefficients are the main focus, thus, connecting mathematical formulation with immediate application in, for example, vibration analysis.

1.3 SPECIFIC OBJECTIVES

The following list of specific objectives has been created for the achievement of the general goal:

- Initial proposal of the method;
- Mathematical formulation of the method for n-th order ODEs;
- Particularization for constant coefficients second order ODEs;

- Solution for non-homogeneous second order ODEs for different classes of excitation functions;
- Mathematical formulation of the method for second order systems of coupled ODEs;
- Particularization for constant matrix coefficients second order ODEs and study of exponential maps;
- Solution for non-homogeneous second order systems of coupled ODEs for different classes of excitation functions;
- Computer implementation of the proposed formulations;
- Mathematical formulation of Heaviside series, *i.e.*, a series of Heaviside step functions for representing discrete or hard to convolute excitation functions, both for one-dimensional and systems of coupled ODEs;
- Numerical experimentation of both accuracy and computational effort for the proposed methods, comparing them to well-established techniques, such as Newmark-beta and State Variables;
- Particularization of the GIF and of the HS methods to systems without classical normal modes.

Other than the above-mentioned topics, two papers with the mathematical formulation have already been submitted to an international journal and one paper on application to Burgers material model for an international conference. A paper has been written for the HS method and is in the final stages before submission to a journal. More works are expected to be written regarding the above developments.

2 ORIGINAL FORMULATION AND APPLICATION FOR ODES WITH A SINGLE DOF

The method that will be proposed in this work is a generalization of the well-established Leibniz integrating factor for first order linear ODEs. It is a widely used method and it is given in the beginning of any undergraduate course in Differential Equations. Thus, as this chapter introduces the proposed technique and its formulation, a brief review of the Leibniz integrating factor will be given and will introduce the intuitions behind the original method and those used to generalize it.

The formulation will be first presented for linear ODEs of n order and with coefficients as function of the independent variable. Thus, the technique will be introduced in the most general form possible. After the initial presentation of the extension, the focus will lay on the advantages of the proposed method and will show applications to common second order linear ODEs that appear in Engineering problems. Some examples are provided, such as Bessel ODE, Euler-Cauchy ODE and with constant coefficients.

Due to the application of second order ODEs with constant coefficients in problems in mechanical vibrations, electric circuitry analysis and constitutive modelling, this case will be particularized for different kinds of excitation functions.

2.1 THE GENERALIZED INTEGRATING FACTOR METHOD

Consider a linear first order ordinary differential equation

$$a_1(t)\dot{y}(t) + a_0(t)y(t) = f(t), \tag{3}$$

where y is the dependent variable, t is the independent variable, $\dot{y}(t)$ is the derivative of y with respect to t and f is a non-homogeneous term, usually known as excitation or source term.

The integrating factor for first order differential equations, introduced by Leibniz, relies on the relation

$$p(t)\dot{y}(t) + \dot{p}(t)y(t) = (p(t)\dot{y}(t)), \tag{4}$$

where $(\dot{\ })$ means the derivative, with respect to t, of all the expression inside the parenthesis.

An integrating factor $\mu(t)$ is multiplied to Eq. (3) to force the appearance of Eq. (4),

$$\underbrace{\mu(t)a_1(t)}_{p(t)}\dot{y}(t) + \underbrace{\mu(t)a_0(t)}_{\dot{p}(t)}y(t) = (p(t)\dot{y}(t)). \tag{5}$$

It immediately follows that

$$\dot{p}(t) = \mu(t)a_0(t) = (\mu(t)\dot{a}_1(t)) = \dot{\mu}(t)a_1(t) + \mu(t)\dot{a}_1(t) \tag{6}$$

such that

$$\frac{\dot{\mu}(t)}{\mu(t)} = \frac{a_0(t) - \dot{a}_1(t)}{a_1(t)}, \tag{7}$$

with solution

$$\mu(t) = \exp\left(\int \frac{a_0(t) - \dot{a}_1(t)}{a_1(t)}\,\mathrm{d}t\right), \tag{8}$$

allowing the use direct integration to solve

$$(\mu(t)a_1(t)y(t))^{\dot{}} = \mu(t)f(t), \tag{9}$$

such that

$$y(t) = \frac{1}{\mu(t)a_1(t)}\left(\int \mu(t)f(t)\,\mathrm{d}t + C_1\right), \tag{10}$$

where C_1 is an integration constant.

Relation given by Eq. (4) can be generalized to higher orders

$$p_j(t)\overset{(j)}{y}(t) + \dot{p}_j(t)\overset{(j-1)}{y}(t) = (p_j(t)\overset{(j-1)}{y}(t))^{\dot{}}, \tag{11}$$

where $\overset{(j)}{y}(t)$ is the j-th derivative of y with respect to t (used for $j > 3$).

Now consider a linear and non-homogeneous ordinary differential equation of order n

$$a_n(t)\overset{(n)}{y}(t) + a_{n-1}(t)\overset{(n-1)}{y}(t) + \cdots + a_2(t)\ddot{y}(t) + a_1(t)\dot{y}(t) + a_0(t)y(t) = f(t). \tag{12}$$

It can be rewritten in pairs of derivatives of $y(t)$ such that one element is a derivative higher than the other. This is achieved by partitioning the coefficients multiplying the intermediate derivatives into two terms

$$\begin{aligned}\underbrace{a_n(t)\overset{(n)}{y}(t) + f_{n,n-1,1}(t)\overset{(n-1)}{y}(t)}_{\pi_{n,n}} + \underbrace{f_{n,n-1,2}(t)\overset{(n-1)}{y}(t) + f_{n,n-2,1}(t)\overset{(n-2)}{y}(t)}_{\pi_{n,n-1}} + \ldots \\ + \underbrace{f_{n,3,2}(t)\dddot{y}(t) + f_{n,2,1}(t)\ddot{y}(t)}_{\pi_{n,3}} + \underbrace{f_{n,2,2}(t)\ddot{y}(t) + f_{n,1,1}(t)\dot{y}(t)}_{\pi_{n,2}} + \\ \underbrace{f_{n,1,2}(t)\dot{y}(t) + a_0(t)y(t)}_{\pi_{n,1}} = f(t),\end{aligned} \tag{13}$$

where $\pi_{n,j}$ is the j-th *partition* of Eq. (12), which is the key idea for the proposed approach. Coefficients $f_{n,j,i}$ refer to the order of the differential equation, n, partition j and $i = 1$ or $i = 2$ such that

$$a_{n,j}(t) = f_{n,j,1}(t) + f_{n,j,2}(t), \tag{14}$$

and there are $N_{\pi_n} = 2n-2$ partitions j.

Multiplying Eq. (13) by a generalized integrating factor $\mu_n(t)$ results in

$$\underbrace{\mu_n(t)a_n(t)\overset{(n)}{y}(t) + \mu_n(t)f_{n,n-1,1}(t)\overset{(n-1)}{y}(t)}_{\left(p_{n,n}\overset{(n-1)}{y}(t)\right)^{\cdot}}$$

$$+\underbrace{\mu_n(t)f_{n,n-1,2}(t)\overset{(n-1)}{y}(t) + \mu_n(t)f_{n,n-2,1}(t)\overset{(n-2)}{y}(t)}_{\left(p_{n,n-1}\overset{(n-2)}{y}(t)\right)^{\cdot}} + \dots$$

$$+\underbrace{\mu_n(t)f_{n,3,2}(t)\dddot{y}(t) + \mu_n(t)f_{n,2,1}(t)\ddot{y}(t)}_{\left(p_{n,3}\ddot{y}(t)\right)^{\cdot}} + \underbrace{\mu_n(t)f_{n,2,2}(t)\ddot{y}(t) + \mu_n(t)f_{n,1,1}(t)\dot{y}(t)}_{\left(p_{n,2}\dot{y}(t)\right)^{\cdot}} +$$

$$\underbrace{\mu_n(t)f_{n,1,2}(t)\dot{y}(t) + \mu_n(t)a_0(t)y(t)}_{\left(p_{n,1}y(t)\right)^{\cdot}} = \mu_n(t)f(t), \qquad (15)$$

such that

$$\left(p_{n,n}\overset{(n-1)}{y}(t)\right)^{\cdot} + \left(p_{n,n-1}\overset{(n-2)}{y}(t)\right)^{\cdot} + \dots + \left(p_{n,1}y(t)\right)^{\cdot} = \mu_n(t)f(t) \qquad (16)$$

can be exactly integrated to

$$p_{n,n}\overset{(n-1)}{y}(t) + p_{n,n-1}\overset{(n-2)}{y}(t) + \dots + p_{n,1}y(t) = \int \mu_n(t)f(t)\,\mathrm{d}t + C_n \qquad (17)$$

an ordinary differential equation of order $n-1$, where C_n is an integration constant.

Coefficients $f_{n,j,i}$ can be found by solving

$$\left(\frac{\dot{\mu}_n}{\mu_n}\right)_n = \left(\frac{\dot{\mu}_n}{\mu_n}\right)_{n-1} = \dots = \left(\frac{\dot{\mu}_n}{\mu_n}\right)_3 = \left(\frac{\dot{\mu}_n}{\mu_n}\right)_2 = \left(\frac{\dot{\mu}_n}{\mu_n}\right)_1. \qquad (18)$$

where

$$\pi_{n,j} \implies \left(\frac{\dot{\mu}_n(t)}{\mu_n(t)}\right)_j = \frac{f_{n,j-1,1}(t) - \dot{f}_{n,j,2}(t)}{f_{n,j,2}(t)}; f_{n,0,1}(t) = a_0(t),\ f_{n,n,2}(t) = a_n(t). \qquad (19)$$

It follows that there are $\frac{n(n-1)}{2}$ combinations of these pair-wise equations, along with the $n-1$ equations relating each pair of partitions with their coefficient. Thus, the number of equations available to evaluate coefficients $f_{n,j,i}$, N_{eq}, is

$$N_{eq_n} = \frac{n(n-1)}{2} + n - 1 = \frac{n^2+n-2}{2} \geq N_{\pi_n};\ n \geq 2, \qquad (20)$$

being always bigger or equal than the number of partitions, N_{π_n}, such that there are enough equations to solve the problem (actually, not all equations must be used). After finding all coefficients $f_{n,j,i}$, it is possible to find $\mu_n(t)$ by using Eq. (19) for just one partition $\hat{j}$ (any partition can be used)

$$\left(\frac{\dot{\mu}_n(t)}{\mu_n(t)}\right)_{\hat{j}} = \frac{f_{n,\hat{j}-1,1}(t) - \dot{f}_{n,\hat{j},2}(t)}{f_{n,\hat{j},2}(t)}, \tag{21}$$

such that

$$\mu_n(t) = \exp\left(\int \frac{f_{n,\hat{j}-1,1}(t) - \dot{f}_{n,\hat{j},2}(t)}{f_{n,\hat{j},2}(t)}\,\mathrm{d}t\right). \tag{22}$$

The same procedure depicted above can be carried out successively until reaching a first order equation, where the traditional integrating factor $\mu_1(t)$ can be used.

The main shortcoming of the proposed procedure is the evaluation of Equations (18). The larger the order of the differential equation, the harder is to find the coefficients $f_{n,j,i}$ at each step of the procedure to decrease the order. For general coefficients $a_j(t)$, Eqs. (18) result in a system of Riccati-like differential equations and for constant coefficients a system of quadratic equations.

Nonetheless, for linear second order equations the procedure can lead to very interesting and practical results, as it will be discussed in the rest of this manuscript. In special, for second order ODEs with constant coefficients, the Riccati equation turns into a simple second order algebraic equation.

2.2 LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

Consider a linear second order ordinary differential equation

$$m(t)\ddot{y}(t) + c(t)\dot{y}(t) + k(t)y(t) = f(t), \tag{23}$$

with initial conditions $\dot{y}(0) = v_0$ and $y(0) = u_0$, where $y(t)$ and $f(t)$ are functions of t over $\mathbb{R}$. This equation is of great interest in physical problems, like for example the vibration of a mass-spring-damper subjected to a force. Thus, independent variable t is also referred to time in the rest of this work.

For physical reasons, it is assumed that both $m(t)$ and $k(t) \in \mathbb{R}_{>0}$ and that $c(t) \in \mathbb{R}^+$, $\forall t$. The explicit dependency on t will be suppressed in the following equations to simplify the notation.

Let start by splitting c as

$$\pi_{2,1} \implies c = f_{2,1,1} + f_{2,1,2}, \tag{24}$$

where $f_{2,1,1}$ and $f_{2,1,2}$ are also function of time, but over $\mathbb{C}$, such that

$$\underbrace{m\ddot{y} + f_{2,1,1}\dot{y}}_{\pi_{2,2}} + \underbrace{f_{2,1,2}\dot{y} + ky}_{\pi_{2,1}} = f. \tag{25}$$

Multiplying the ODE by integrating factor $\mu_2(t)$ results in

$$\underbrace{\mu_2 m}_{p_{2,2}}\ddot{y} + \underbrace{\mu_2 f_{2,1,1}}_{\dot{p}_{2,2}}\dot{y} + \underbrace{\mu_2 f_{2,1,2}}_{p_{2,1}}\dot{y} + \underbrace{\mu_2 k}_{\dot{p}_{2,1}}y = \mu_2 f, \tag{26}$$

or

$$p_{2,2}\ddot{y} + \dot{p}_{2,2}\dot{y} + p_{2,1}\dot{y} + \dot{p}_{2,1}y = \mu_2 f. \tag{27}$$

Analysing the term

$$\dot{p}_{2,2} = \mu_2 f_{2,1,1} = \dot{(\mu_2 m)} = \dot{\mu}_2 m + \mu_2\dot{m}, \tag{28}$$

it is possible to state that

$$\frac{\dot{\mu}_2}{\mu_2} = \frac{f_{2,1,1} - \dot{m}}{m}. \tag{29}$$

Following the same procedure,

$$\dot{p}_{2,1} = \mu_2 k = \dot{(\mu_2 f_{2,1,2})} = \dot{\mu}_2 f_{2,1,2} + \mu_2 \dot{f}_{2,1,2} \tag{30}$$

such that

$$\frac{\dot{\mu}_2}{\mu_2} = \frac{k - f_{2,1,2}}{f_{2,1,2}}. \tag{31}$$

Thus, by relating Eqs. (29) and (31), one obtain the particular form of Eq. (18)

$$\frac{f_{2,1,1} - \dot{m}}{m} = \frac{k - f_{2,1,2}}{f_{2,1,2}}. \tag{32}$$

Since $c = f_{2,1,1} + f_{2,1,2}$ it is possible to state that $f_{2,1,1} = c - f_{2,1,2}$. Using Eq. (32)

$$(c - f_{2,1,2} - \dot{m})\, f_{2,1,2} = \left(k - \dot{f}_{2,1,2}\right) m \tag{33}$$

such that

$$f_{2,1,2}^2 = \dot{f}_{2,1,2}m - km + (c - \dot{m})f_{2,1,2}, \tag{34}$$

is a Riccati differential equation [1].

[1] Conversely, it is also possible to define $f_{2,1,2} = c - f_{2,1,1}$ such that

$$\frac{f_{2,1,1} - \dot{m}}{m} = \frac{k - \dot{c} + \dot{f}_{2,1,1}}{c - f_{2,1,1}} \tag{35}$$

or

$$f_{2,1,1}^2 = -\dot{f}_{2,1,1}m + (\dot{m} + c)f_{2,1,1} - \dot{m}c + (\dot{c} - k)m, \tag{36}$$

also a Riccati differential equation.

Equation (31) is a first order ODE with known solution

$$\mu_2 = \exp \int \frac{k - \dot{f}_{2,1,2}}{f_{2,1,2}}\, \mathrm{d}t, \tag{37}$$

where $f_{2,1,2}$ is obtained from Eq. (34). It is important to stress that only the particular solution of the Riccati equation is needed.

By knowing μ_2 it is possible to re-write Eq. (27) as

$$\dot{(p_{2,2}\dot{y})} + \dot{(p_{2,1}y)} = \mu_2 f, \tag{38}$$

such that integrating with respect to t results in

$$(p_{2,2}\dot{y}) + (p_{2,1}y) = \underbrace{\int \mu_2 f\, \mathrm{d}t + C_2}_{h}, \tag{39}$$

a first order ODE. Using another integrating factor μ_1 such that

$$\underbrace{\mu_1 p_{2,2}}_{p_{1,1}}\dot{y} + \underbrace{\mu_1 p_{2,1}}_{\dot{p}_{1,1}}y = \mu_1 h \tag{40}$$

or

$$p_{1,1}\dot{y} + \dot{p}_{1,1}y = \mu_1 h. \tag{41}$$

Following the same procedure

$$\dot{p}_{1,1} = \mu_1 p_{2,1} = \dot{(\mu_1 p_{2,2})} = \dot{\mu}_1 p_{2,2} + \mu_1 \dot{p}_{2,2}, \tag{42}$$

such that

$$\frac{\dot{\mu}_1}{\mu_1} = \frac{p_{2,1} - \dot{p}_{2,2}}{p_{2,2}}, \tag{43}$$

with known solution

$$\mu_1 = \exp \int \frac{p_{2,1} - \dot{p}_{2,2}}{p_{2,2}}\, \mathrm{d}t. \tag{44}$$

Equation (41) can be written as

$$\dot{(p_{1,1}y)} = \mu_1 h, \tag{45}$$

and, integrating with respect to time, results in

$$p_{1,1}y = \int \mu_1 h\, \mathrm{d}t + C_1, \tag{46}$$

such that

$$y = \frac{1}{p_{1,1}} \int \mu_1 h\, \mathrm{d}t + \frac{1}{p_{1,1}} C_1. \tag{47}$$

Thus, by using the definition of both $p_{1,1}$ and h

$$y(t)=\frac{1}{\mu_1(t)\mu_2(t)m(t)}\left(\int\mu_1(t)\left(\int\mu_2(t)f(t)\,\mathrm{d}t\right)\mathrm{d}t+\int\mu_1(t)C_2\,\mathrm{d}t+C_1\right) \tag{48}$$

is the general solution for the second order ordinary differential equation stated in Eq. (23). This general solution can be split into its particular, $y_p(t)$, and homogeneous, $y_h(t)$ parts

$$y_p(t)=\frac{1}{\mu_1(t)\mu_2(t)m(t)}\int\mu_1(t)\int\mu_2(t)f(t)\,\mathrm{d}t\,\mathrm{d}t \tag{49}$$

and

$$y_h(t)=\frac{1}{\mu_1(t)\mu_2(t)m(t)}\left(\int\mu_1(t)C_2\,\mathrm{d}t+C_1\right), \tag{50}$$

such that $y(t)=y_p(t)+y_h(t)$. Constants C_1 and C_2 can be found by considering the solution at known t values.

Thus, the proposed solution depends on the solution of a Riccati equation and always results in integrating factors $\mu_1(t)$ and $\mu_2(t)$ function of t only, disregarding the form of the coefficients $m(t)$, $c(t)$ and $k(t)$.

Solution of Eq. (34) is fundamental for the success of the proposed formulation. Indeed, the solution of the Riccati equation is not an easy task if general coefficients $m(t)$, $c(t)$ and $k(t)$ are considered. Nonetheless, only the particular solution is needed. Various analytical solutions can be found in the literature for specific forms of coefficients (HARKO; LOBO; MAK, 2014), as well as numerical methods (FILE; AGA, 2016). Riccati equations with polynomial coefficients also present interesting solution properties (CAMPBELL; GOLOMB, 1954; NAVICKAS et al., 2017). The Kudryashov method and its versions can be used to derive analytical solutions to many nonlinear differential equations starting with the Riccati equation (KUDRYASHOV, 2003; KILICMAN; SILAMBARASAN, 2018; GABER et al., 2019; KAPLAN; BEKIR; AKBULUT, 2016). A more detailed discussion on the solution of the Riccati equation is out of the scope of this text, since the main focus in on the solution of linear second order ODEs with constant coefficients, where the Riccati equation turns into a simple algebraic equation with direct solution. Nonetheless, some particular solutions are developed in A.1 to aid in the solution of the following examples. Other interesting conditions to obtain the particular solution for the Riccati equation are developed in Appendix A.2.

Appendix A.2 can be used to derive particular solutions to many different and difficult problems, which might have poles and singular points. Thus, the solution of the Riccati differential equation can be easily found and, consequently, enables the generalized integrating factor to overcome linear second order differential equations with singular points, that would be a major drawback for numerical methods and for series methods, due to limited radius of convergence, (BOYCE; DIPRIMA, 2001). Intuitions in Appendix A.2 might also help developing new numerical and approximate methods to solve Riccati differential equations and, hence, linear second order equations due to the generalized integrating factor.

Example I - Cauchy-Euler equation

Consider the second order ODE

$$t^2\ddot{y}(t) - 2t\dot{y}(t) + 2y(t) = 6t^2 + 4\ln(t), \tag{51}$$

such that the coefficients are $m(t) = t^2$, $c(t) = -2t$ and $k(t) = 2$ and the excitation is $f(t) = 6t^2 + 4\ln(t)$.

The Riccati equation, Eq. (34), can be written as

$$f_{2,1,2}^2 = t^2\dot{f}_{2,1,2} - 2t^2 - 4tf_{2,1,2}, \tag{52}$$

which is case Eq. (618) in Appendix A.2, resulting in the particular solution $f_{2,1,2} = -t$. Using Eq. (37)

$$\mu_2 = \exp\int \frac{2+1}{-t}\,\mathrm{d}t = \frac{1}{t^3}. \tag{53}$$

Thus, $p_{2,2} = \mu_2 m = t^{-1}$ and $p_{2,1} = \mu_2 f_{1,2} = -t^{-2}$. Using Eq. (44)

$$\mu_1 = \exp\int \frac{-t^{-1}+t^{-2}}{t^{-1}}\,\mathrm{d}t = 1. \tag{54}$$

Complete solution is given by Eq. (48)

$$y(t) = \frac{1}{1t^{-3}t^2}\left(\int_0^t 1\left\{\int_0^t t^{-3}\left(6t^2+4\ln(t)\right)\mathrm{d}t\right\}\mathrm{d}t + \int_0^t 1C_2\,\mathrm{d}t + C_1\right) \tag{55}$$

such that

$$y(t) = (6t^2+2)\ln(t) - 6t^2 + 3 + C_2t^2 + C_1t, \tag{56}$$

the correct analytical solution.

Example II - Bessel equation

Consider ODE

$$t^2\ddot{y}(t) + t\dot{y}(t) + \left(t^2 - \frac{1}{4}\right)y(t) = f(t) = t^{\frac{3}{2}}, \tag{57}$$

with known values $y(t_0) = u_0 = 0$ and $\dot{y}(t_0) = v_0 = 0$ at $t_0 = 0.1$. It is known that solution to this equation is singular at $t = 0$.

The corresponding Riccati equation, Eq. (34), is

$$f_{2,1,2}^2 = t^2\dot{f}_{2,1,2} - tf_{2,1,2} + \frac{t^2}{4} - t^4, \tag{58}$$

whose candidate particular solution is a second order polynomial

$$\tilde{f}_{2,1,2} = z_0 + z_1 t + z_2 t^2. \tag{59}$$

Applying this polynomial into Eq. (58) yields

$$z_0^2 + 2z_0 z_1 t + \left(2z_0 z_2 + z_1^2\right) t^2 + 2z_1 z_2 t^3 + z_2^2 t^4 = z_1 t^2 + 2z_2 t^3 - z_0 t - z_1 t^2 - z_2 t^3 + \frac{t^2}{4} - t^4, \tag{60}$$

that simplifies to

$$z_0^2 + 2z_0 z_1 t + \left(2z_0 z_2 + z_1^2\right) t^2 + 2z_1 z_2 t^3 + z_2^2 t^4 = -z_0 t + \frac{t^2}{4} + z_2 t^3 - t^4, \tag{61}$$

whose solution is $z_0 = 0$, $z_1 = \frac{1}{2}$ and $z_2 = i$, which is the case of Eq. (618) in Appendix A.2.

Thus, the integrating factor, Eq. (37), is

$$\mu_2 = \exp\left(\int \frac{t^2 - \frac{1}{4} - \frac{1}{2} - 2it}{\frac{t}{2} + it^2} \,\mathrm{d}t\right) = \exp\left(\int \frac{t^2 - 2it - \frac{3}{4}}{\frac{t}{2} + it^2} \,\mathrm{d}t\right). \tag{62}$$

The polynomials in the integrand can be factored and simplified to

$$\mu_2 = \exp\left(\int \frac{\left(t - \frac{i}{2}\right)\left(t - \frac{3i}{2}\right)}{it\left(t - \frac{i}{2}\right)} \,\mathrm{d}t\right) = \exp\left(\int -i \,\mathrm{d}t - \frac{3}{2}\int \frac{1}{t} \,\mathrm{d}t\right), \tag{63}$$

such that

$$\mu_2 = \exp\left(-it - \frac{3}{2} ln|t|\right) = t^{-\frac{3}{2}} \exp(-it). \tag{64}$$

The integrating factor to integrate the differential equation from first order to an algebraic equation is evaluated using Eq. (44),

$$\mu_1 = t^{\frac{1}{2}} \exp(it). \tag{65}$$

Solution is then given by Eq. (48)

$$y(t) = t^{-\frac{1}{2}} \exp(-it) \int \exp(2it) \int t^{-\frac{3}{2}} \exp(-t) f(t) \,\mathrm{d}t \,\mathrm{d}t + C_1 t^{-\frac{1}{2}} \exp(it) + C_2 t^{-\frac{1}{2}} \exp(-it), \tag{66}$$

and replacing the expression for $f(t)$

$$y(t) = t^{-\frac{1}{2}} + C_1 t^{-\frac{1}{2}} \exp(it) + C_2 t^{-\frac{1}{2}} \exp(-it). \tag{67}$$

Constants, C_1 and C_2, can be obtained by solving a system of linear equations

$$\begin{bmatrix} t_0^{-\frac{1}{2}} \exp(it_0) & t_0^{-\frac{1}{2}} \exp(-it_0) \\ \left(it_0^{-\frac{1}{2}} - \frac{t_0^{-\frac{3}{2}}}{2}\right) \exp(it_0) & -\left(it_0^{-\frac{1}{2}} + \frac{t_0^{-\frac{3}{2}}}{2}\right) \exp(-it_0) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} u_0 - t_0^{-\frac{1}{2}} \\ v_0 + \frac{t_0^{-\frac{3}{2}}}{2} \end{Bmatrix}. \tag{68}$$

Using values at $t_0 = 0.1$ results in

$$y(t) = t^{-\frac{1}{2}} + (-0.4975 + 0.04992i)\, t^{-\frac{1}{2}} \exp(it) - (0.4975 + 0.04992i)\, t^{-\frac{1}{2}} \exp(-it), \tag{69}$$

or, by using the Euler identity[2],

$$y(t) = \frac{(1 - 0.995\cos(t) - 0.09984\sin(t))}{\sqrt{t}}. \tag{71}$$

Figure 1 compares the real part of the solution obtained by the proposed approach, Eq. (71), with the solution obtained by using the Tsitouras $5/4$ Runge-Kutta method (TSITOURAS, 2011) with adaptive time step. The numerical solution starts at $t = 0.1$ (dark dotted line) due to the singularity but the analytical solution properly captures the singularity at $t \to 0$ (solid blue curve).

2

$$sin(\omega t + \phi) = \frac{i}{2}\left(e^{-i(\omega t+\phi)} - e^{i(\omega t+\phi)}\right). \tag{70}$$

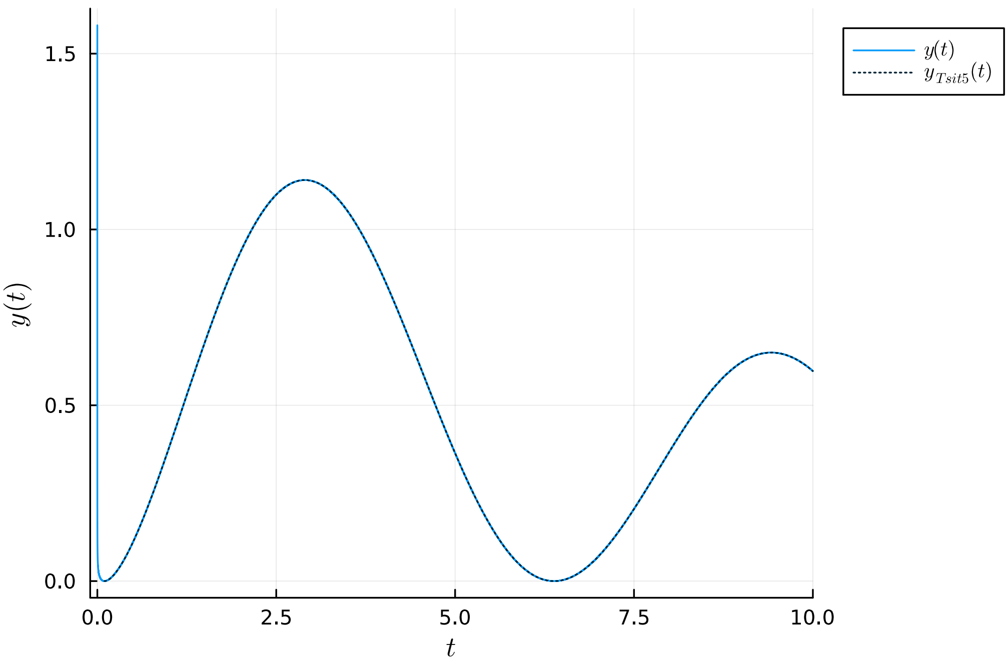

Figure 1 – Analytical response $y(t)$ obtained for the Bessel equation, Eq. (57), and the solution obtained by using a numerical method, $y_{Tsit5}(t)$.

2.3 SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Previous equations are much simpler when coefficients m, c and k are constant since $\dot{m}$, $\dot{c}$, $\dot{f}_{2,1,2}$ and $\dot{k}$ are zero. As $m > 0$, it is possible to normalize the equation by m such that

$$\ddot{y}(t) + \bar{c}\dot{y}(t) + \bar{k}y(t) = \bar{f}(t), \tag{72}$$

where $\bar{c} = c/m$, $\bar{k} = k/m$ and $\bar{f} = f/m$. It follows that Eq. (34) for the constant coefficients case falls into Eq. (608), thus, it reduces to an algebraic quadratic equation

$$f_{2,1,2}^2 = -\bar{k} + \bar{c}f_{2,1,2} \tag{73}$$

with **direct solution**

$$f_{2,1,2} = \frac{\bar{c} \pm \sqrt{\bar{c}^2 - 4\bar{k}}}{2}, \tag{74}$$

a complex number when $\bar{c}^2 - 4\bar{k} < 0$ (under damped problems). Any one of the two roots can be used. Equation (37) reduces to

$$\mu_2 = \exp\left(\hat{k}t\right) \tag{75}$$

where

$$\hat{k} = \frac{\bar{k}}{f_{2,1,2}} \tag{76}$$

and Eq. (44) to

$$\mu_1 = \frac{\exp\left(f_{2,1,2}t\right)}{\exp\left(\hat{k}t\right)} = \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right). \tag{77}$$

Thus, the complete solution is

$$y(t) = \frac{1}{\mu_1(t)\mu_2(t)}\left(\int \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\left(\int \exp\left(\hat{k}t\right)\bar{f}(t)\,\mathrm{d}t + C_2\right)\mathrm{d}t + C_1\right) \tag{78}$$

where the term

$$\frac{1}{\mu_1(t)\mu_2(t)} = \exp\left(-f_{2,1,2}t\right) \tag{79}$$

such that

$$y(t) = \exp\left(-f_{2,1,2}t\right)\left(\int \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\left(\int \exp\left(\hat{k}t\right)\bar{f}(t)\,\mathrm{d}t + C_2\right)\mathrm{d}t + C_1\right), \tag{80}$$

is the complete solution. Additionally, it is possible to split the solution in its particular

$$y_p(t) = \exp\left(-f_{2,1,2}t\right)\left(\int \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\left(\int \exp\left(\hat{k}t\right)\bar{f}(t)\,\mathrm{d}t\right)\mathrm{d}t\right), \tag{81}$$

and homogeneous parts

$$y_h(t) = \exp\left(-f_{2,1,2}t\right)\left(\int \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)C_2\,\mathrm{d}t + C_1\right). \tag{82}$$

If $\bar{k} \neq 0$, which is a fair and physical assumption, there are two cases: if $f_{2,1,2}-\hat{k}=0$ and otherwise. From Eq. (74), one realizes that the first case occurs when damping is critical. When $f_{2,1,2}-\hat{k}=0$, Equation (82) simplifies to

$$y_h(t) = C_2 t\exp\left(-f_{2,1,2}t\right) + C_1\exp\left(-f_{2,1,2}t\right). \tag{83}$$

This result is what one would expect for critical damping (BOYCE; DIPRIMA, 2001). However, no solution assumption was made, nor the Wronskian was used to prove the linear independence of the new solution multiplied by t, as it is done in (BOYCE; DIPRIMA, 2001). In contrast, if $f_{2,1,2}-\hat{k}\neq 0$, Equation (82) is further simplified to

$$y_h(t) = \exp\left(-f_{2,1,2}t\right)C_1 + C_2\exp\left(-\hat{k}t\right). \tag{84}$$

Particular solution for the constant coefficient case, Eq. (81), can be further developed if some particular form for $\bar{f}(t)$ is assumed. The following sections are devoted to investigate two situations: Continuous excitation (periodic and polynomial) and discontinuous (unitary impulse and Heaviside). The case of a general function multiplied by Heavisides is used to obtain the analytical solution to complicate loading scenarios. One might conclude that Eq. (81) will also have different treatment when $f_{2,1,2}-\hat{k}=0$, but that is not necessarily true. Thus, for each one of the excitation functions explored below, a comment on the need and reason for different cases of integration will be pointed out.

2.4 SECOND ORDER ODES WITH CONSTANT COEFFICIENTS - CONTINUOUS EXCITATION FUNCTIONS

2.4.1 Periodic excitation

Lets consider a further hypothesis: excitation $f(t)$ is periodic or its Fourier series is convergent. Thus, normalized excitation $\bar{f}(t)$ can be represented as a series of exponentials

$$\bar{f}(t) = \sum_{k=1}^{n_k} c_k \exp(\beta_k t + \phi_k), \tag{85}$$

where n_k is the number of terms, $c_k \in \mathbb{R}$ is an amplitude, $\beta_k = i\omega_k \in \mathbb{C}$ is a complex angular frequency and $\phi_k \in \mathbb{C}$ is a complex phase.

Previously, it was seen that the integrating factor is an exponential by definition, and that the particular solution, *i.e.*, the solution corresponding to the excitation function, naturally appears through the successive integrations. These integrations are convolutions over the integrating factor, thus, if the excitation can be expressed in terms of exponentials, these convolutions can be trivially calculated. Applying Eq. (85) into Eq. (81) yields

$$y_p(t) = \exp(-f_{2,1,2}t) \int \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int \exp\left(\hat{k}t\right) \sum_{k=1}^{n_k} c_k \exp(\beta_k t + \phi_k) \,\mathrm{d}t \right) \mathrm{d}t, \tag{86}$$

using the multiplication property between exponentials

$$y_p(t) = \exp(-f_{2,1,2}t) \int \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int \sum_{k=1}^{n_k} c_k \exp\left(\left(\beta_k + \hat{k}\right) t + \phi_k\right) \mathrm{d}t \right) \mathrm{d}t, \tag{87}$$

whose inner integral is trivial and results in

$$y_p(t) = \exp(-f_{2,1,2}t) \int \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \sum_{k=1}^{n_k} \frac{c_k}{\beta_k + \hat{k}} \exp\left(\left(\beta_k + \hat{k}\right) t + \phi_k\right) \mathrm{d}t. \tag{88}$$

If $f_{2,1,2} - \hat{k} = 0$, there is no need to integrate in different fashion, since the function that multiplies the exponential with the null exponent is also an exponential function, hence, they can simply have their exponents summed up. Again, rearranging the multiplication of exponentials, yields

$$y_p(t) = \exp(-f_{2,1,2}t) \int \sum_{k=1}^{n_k} \frac{c_k}{\beta_k + \hat{k}} \exp((\beta_k + f_{2,1,2}) t + \phi_k) \,\mathrm{d}t, \tag{89}$$

which is again trivially integrated to

$$y_p(t) = \exp(-f_{2,1,2}t) \sum_{k=1}^{n_k} \frac{c_k}{\left(\beta_k + \hat{k}\right)(\beta_k + f_{2,1,2})} \exp((\beta_k + f_{2,1,2}) t + \phi_k), \tag{90}$$

such that

$$y_p(t) = \sum_{k=1}^{n_k} \frac{c_k \exp(\beta_k t + \phi_k)}{(\beta_k + \hat{k})(\beta_k + f_{2,1,2})}. \tag{91}$$

Equation (91) gives a closed-form and analytic particular solution for any periodic excitation or an excitation represented through its Fourier series expansion. It is worth noticing that all constants given by the two integrations at time $t_0 = 0$ were omitted for they can be coupled to the constants C_1 and C_2, which are by definition constants of these very integrations.

Example with periodic excitation

Consider a mechanical system, made out of elements of stiffness, mass and damping, subjected to periodic loads. Despite the quantity of those elements, if all the movements can be kinetically determined in terms of a single movement, the whole phenomenon is described by a single degree of freedom second order differential equation. In most applications of vibration problems, the load is periodic, like in support excitation and in unbalanced or electromagnetic machines.

Consider a mechanical system with an equivalent mass of 1 kg, an equivalent damping coefficient of 2 Ns/m, and an equivalent stiffness of 10 N/m. The ODE representing the dynamic equilibrium of this system is given by

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = \bar{f}(t), \tag{92}$$

and non-homogeneous initial conditions $u_0 = 0.2$ and $v_0 = 0.0$ at $t_0 = 0$ are assumed. The periodic excitation force is given by

$$\bar{f}(t) = -cos(0.5t) + sin(t) + cos(1.5t - 1.5) - 2sin(2t) + 2sin(10t) \tag{93}$$

the superposition of many different signals, like it would be expected in real world problems. This loading can be written as an exponential series by using the Euler identity

$$\bar{f}(t) = -\frac{1}{2}e^{0.5it} - \frac{1}{2}e^{-0.5it} - \frac{i}{2}e^{it} + \frac{i}{2}e^{-it} + \frac{1}{2}e^{1.5i(t-1)} + \frac{1}{2}e^{1.5i(1-t)} + ie^{2it} - ie^{-2it} - ie^{10it} + ie^{-10it} = \sum_{j=1}^{10} c_j e^{\beta_j t + \phi_j}. \tag{94}$$

For this example, one gets

$$\bar{k} = 10, \tag{95}$$

$$f_{2,1,2} = 1 + 3i, \tag{96}$$

$$\hat{k} = 1 - 3i, \tag{97}$$

Particular solution can be found using Eq. (91)

$$y_p(t) = \frac{-78+8i}{1537}e^{0.5it} - \frac{78+8i}{1537}e^{-0.5it} - \frac{2+9i}{170}e^{it} + \frac{-2+9i}{170}e^{it} + \frac{62-24i}{1105}e^{1.5i(t-1)} + \frac{62+24i}{1105}e^{1.5i(1-t)} + \frac{2+3i}{26}e^{2it} + \frac{2-3i}{26}e^{-2it} + \frac{-2+9i}{850}e^{10it} - \frac{2+9i}{850}e^{-10it}, \tag{98}$$

and, since $f_{2,1,2} - \hat{k} \neq 0$, the homogeneous solution is given by Eq. (84)

$$y_h(t) = C_2 e^{(-1+3i)t} + C_1 e^{-(1+3i)t}. \tag{99}$$

Evaluation of integration constants C_1 and C_2 is straightforward. Using Eq. (98) and Eq. (99) at $t = 0$, results in the complex conjugates, $C_2 \approx 0.10564 - 0.11980i$ and $C_1 \approx 0.10564 + 0.11980i$.

Figure 2 shows the real part of the homogeneous solution, $y_h(t)$ (solid blue line), the real part of the permanent solution (solid read line) and the real part of the complete solution (solid green line) obtained using the proposed approach. The complete solution obtained with the Newmark-beta method, $\tilde{y}(t)$, with $\Delta t = 0.001$s is shown as the dark dotted line. It is possible to infer that the numerical solution matches the complete solution obtained with the proposed approach.

It is worth mentioning that although $y(t)$ is complex, the maximum amplitude of the complex part of the response was 8.3267×10^{-17} in the analysed time interval, which is zero when compared to the real part. The same pattern was observed in all examples studied in this chapter.

Since it is known that the response to this problem is real it is worth showing that this is indeed just a matter of representation. To this end, Eqs. (98) and (99) can be transformed to a real-valued form by applying the Euler identity in reverse, resulting in

$$y_p(t) = -\frac{156}{1537}cos(0.5t) - \frac{16}{1537}sin(0.5t) - \frac{2}{85}cos(t) + \frac{9}{85}sin(t) + \frac{124}{1105}cos(1.5(t-1)) + \frac{48}{1105}sin(1.5(t-1)) + \frac{2}{13}cos(2t) - \frac{3}{13}sin(2t) - \frac{2}{425}cos(10t) - \frac{9}{425}sin(10t), \tag{100}$$

and

$$y_h(t) = C_2 e^{-t} cos(3t) + C_1 e^{-t} sin(3t), \tag{101}$$

with $C_2 \approx 0.2107$ and $C_1 \approx 0.0702$. Complete solution $y(t) = y_p(t) + y_h(t)$ is correct and satisfies Eq. (92) for all t.

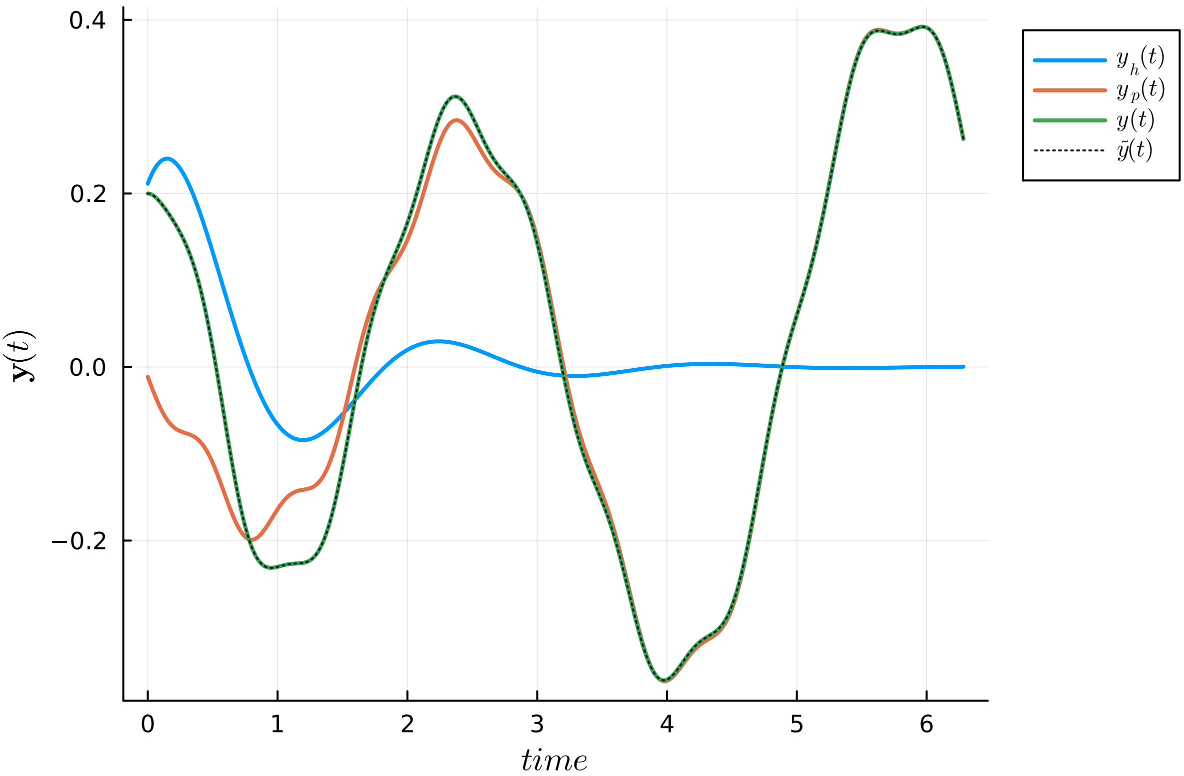

Figure 2 – Homogeneous solution, $y_h(t)$ (solid blue line), permanent solution $y_p(t)$ (solid red line) and complete solution $y(t)$ (solid green line) obtained with the proposed approach for periodic loading, Eq. (93). Complete solution obtained using the Newmark-beta method for $\Delta t = 0.001$ s, $\tilde{y}(t)$, is shown as a dark dotted line.

2.4.2 Polynomial excitation

Polynomial excitations can be used, for example, to model strain in a viscoelastic testing sample, since polynomials are smooth functions and can be tailored to match specific shapes and paths. Thus, finding analytical solutions to this type of excitation can be useful for determining properties of the viscoelastic medium using such kind of experiments. Polynomial excitations are also commonly found in RCL circuits.

Assuming a polynomial excitation in the form

$$\bar{f}(t) = \sum_{k=0}^{n_p} c_k \left(t - t_s\right)^k, \tag{102}$$

where $c_k \in \mathbb{R}$ are coefficients, n_p is the number of terms and $t_s \in \mathbb{R}$ a time shift. Applying Eq. (102) into Eq. (81) yields

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \int \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int \exp\left(\hat{k}t\right) \sum_{k=0}^{n_p} c_k \left(t - t_s\right)^k \mathrm{d}t \right) \mathrm{d}t, \tag{103}$$

as integration is a linear operator, the summation can be transferred to the whole integration and the coefficient of each power of t can also be put out of the integral,

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \sum_{k=0}^{n_p} c_k \int \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right) \left(\int \exp\left(\hat{k}t\right)(t-t_s)^k \, \mathrm{d}t\right) \mathrm{d}t. \tag{104}$$

The convolution of a polynomial over an exponential is recursively evaluated using integration by parts. For it, let the power of t be a positive integer α. The first integration by parts is

$$\int_0^t \exp(\beta t)(t-t_s)^\alpha \, \mathrm{d}t = \int_0^t \left(\frac{1}{\beta}\exp(\beta t)\right)^{\cdot} (t-t_s)^\alpha \, \mathrm{d}t = \frac{1}{\beta}\exp(\beta t)(t-t_s)^\alpha \Big|_0^t - \frac{1}{\beta}\alpha \int \exp(\beta t)(t-t_s)^{\alpha-1} \, \mathrm{d}t, \tag{105}$$

such that another convolution with a smaller power appeares in the RHS of Eq. (105). This procedure can be used recursively until the null power, where the integral is over the exponential only,

$$\begin{aligned}\int_0^t \exp(\beta t)(t-t_s)^\alpha \, \mathrm{d}t &= \frac{1}{\beta}(t-t_s)^\alpha \exp(\beta t)\Big|_0^t \\ &- \left(\frac{1}{\beta}\right)^2 \alpha (t-t_s)^{\alpha-1}\exp(\beta t)\Big|_0^t + \left(\frac{1}{\beta}\right)^3 \alpha(\alpha-1)(t-t_s)^{\alpha-2}\exp(\beta t)\Big|_0^t \\ &- \left(\frac{1}{\beta}\right)^4 \alpha(\alpha-1)(\alpha-2)(t-t_s)^{\alpha-3}\exp(\beta t)\Big|_0^t \ldots \\ &+ (-1)^\alpha \left(\frac{1}{\beta}\right)^\alpha \alpha! \int \exp(\beta t)(t-t_s)^{\alpha-\alpha} \, \mathrm{d}t.\end{aligned} \tag{106}$$

Rarranging the terms,

$$\begin{aligned}\int_0^t \exp(\beta t)(t-t_s)^\alpha \, \mathrm{d}t &= \\ \sum_{l=1}^{\alpha} (-1)^{l+1} \left(\frac{1}{\beta}\right)^l \frac{\alpha!}{(\alpha-l+1)!}(t-t_s)^{\alpha-l+1}\exp(\beta t)\Big|_0^t \\ &+ (-1)^\alpha \left(\frac{1}{\beta}\right)^\alpha \alpha! \exp(\beta t)\left(\frac{1}{\beta}\right),\end{aligned} \tag{107}$$

which can be further simplified to

$$\begin{aligned}\int_0^t \exp(\beta t)(t-t_s)^\alpha \, \mathrm{d}t &= \\ \sum_{l=1}^{\alpha+1} (-1)^{l+1} \left(\frac{1}{\beta}\right)^l \frac{\alpha!}{(\alpha-l+1)!}(t-t_s)^{\alpha-l+1}\exp(\beta t)\Big|_0^t.\end{aligned} \tag{108}$$

Applying this result to Eq. (104) and neglecting the evaluation of this integral at $t = 0$, since it can be summed with constant C_1, yields

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \sum_{k=0}^{n_p} c_k \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \sum_{l=1}^{k+1} (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{(k-l+1)!} (t - t_s)^{k-l+1} \exp\left(\hat{k}t\right) \mathrm{d}t. \tag{109}$$

Again, for the same reason as for the periodic excitation case, if $f_{2,1,2} - \hat{k} = 0$, it can be convoluted just as it is. Thus, rearranging terms, using the linearity of the integration operator and defining $r = k - l + 1$

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \sum_{k=0}^{n_p} c_k \sum_{l=1}^{k+1} (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{r!} \int_0^t \exp\left(f_{2,1,2}t\right) (t - t_s)^r \,\mathrm{d}t. \tag{110}$$

Using the result from Eq. (108),

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \sum_{k=0}^{n_p} c_k \sum_{l=1}^{k+1} (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{(r)!} \sum_{p=1}^{r+1} (-1)^{p+1} f_{2,1,2}^{-p} \frac{r!}{(r+1-p)!} (t - t_s)^{r+1-p} \exp\left(f_{2,1,2}t\right), \tag{111}$$

that can be further simplified to

$$y_p(t) = \sum_{k=0}^{n_p} c_k \sum_{l=1}^{k+1} (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{r!} \sum_{p=1}^{r+1} (-1)^{p+1} f_{2,1,2}^{-p} \frac{r!}{(r+1-p)!} (t - t_s)^{r+1-p}, \tag{112}$$

which is itself another polynomial.

Example with polynomial excitation

RLC circuits consist of resistors, inductors and capacitors, and are present in many electrical and electronic applications, such as filters, control and even health-purposed circuits like pacemakers (AGARWAL; LANG, 2005; IRWIN; NELMS, 2006). Various excitation functions in electric circuitry problems can be modelled by polynomials, at least locally, and the ramp function is a traditional example (a first order polynomial).

In this example, the ramp function has unitary slope,

$$\bar{f}(t) = t, \tag{113}$$

and the circuit has resistance of 0.5 Ω, capacitance of 1 F, and inductance of 0.1 H. Thus, the circuit differential equation is

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = t, \tag{114}$$

with $t \in [0,4]$, and homogeneous initial conditions $u_0 = 0$ and $v_0 = 0$ at $t_0 = 0$.

Steps used to solve Eq. (112), $y_p(t)$, are depicted in an algorithm form in Alg. 2, resulting in $y_p(t) = 0.1t - 0.02$. Since $\bar{c} - 4\bar{k} = -36 \implies f_{2,1,2} - \hat{k} \neq 0$, the homogeneous solution is given by Eq. (84), yielding a linear system of equations for its integration constants, C_1 and C_2,

$$\begin{bmatrix} \exp(0) & \exp(0) \\ -f_{2,1,2}\exp(0) & -\hat{k}\exp(0) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} u_0 - y_p(0) \\ v_0 - \dot{y}_p(0) \end{Bmatrix}, \tag{115}$$

which simplifies to

$$\begin{bmatrix} 1 & 1 \\ -1-3i & -1+3i \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0.02 \\ -0.1 \end{Bmatrix}, \tag{116}$$

whose solution is $C_1 = 0.0100 - 0.0133i$ and $C_2 = 0.0100 + 0.0133i$. Thus, the complete solution is

$$\begin{aligned} y(t) = 0.1t - 0.02 + (0.0100 - 0.0133i)\exp(-(1+3i)t) + \\ (0.0100 + 0.0133i)\exp(-(1-3i)t), \end{aligned} \tag{117}$$

or, by using the Euler identity,

$$y(t) = 0.1t - 0.0266\exp(-t)\sin(3t) + 0.02\exp(-t)\cos(3t) - 0.02. \tag{118}$$

Figure 3 shows (the real part) of the homogeneous solution $y_h(t)$ (solid blue line), the permanent solution $y_p(t)$ (solid red line) and the complete solution $y(t)$ (solid green line) obtained with the proposed approach for the polynomial excitation (linear ramp). The complete solution obtained with the Newmark-beta method $\tilde{y}(t)$ (dotted line) for $\Delta t = 0.001$s matches the complete solutions obtained with the proposed approach. The complex part of the solution is negligible when compared to the real part (numerically zero) for all t.

Algorithm 1: Evaluation of Eq. (112)

$k = 0$

$c_0 = 0$

$k = 1$

$c_1 = 1$

$l = 1 \implies (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{r!} = 0.1 + 0.3i$

$p = 1 \implies (-1)^{p+1} f_{2,1,2}^{-p} \frac{r!}{(r+1-p)!} (t - t_s)^{r+1-p} = (0.1 - 0.3i)\, t$

$p = 2 \implies (-1)^{p+1} f_{2,1,2}^{-p} \frac{r!}{(r+1-p)!} (t - t_s)^{r+1-p} = 0.08 + 0.06i$

$l = 2 \implies (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{r!} = 0.08 - 0.06i$

$p = 1 \implies (-1)^{p+1} f_{2,1,2}^{-p} \frac{r!}{(r+1-p)!} (t - t_s)^{r+1-p} = 0.1 - 0.3i$

$y_p = (1\,(0.1 + 0.3i)\,(0.1 - 0.3i)\,t) + (1\,(0.1 + 0.3i)\,(0.08 + 0.06i)) +$
$(1\,(0.08 - 0.06i)\,(0.1 - 0.3i))$
$y_p = 0.1t - 0.02$

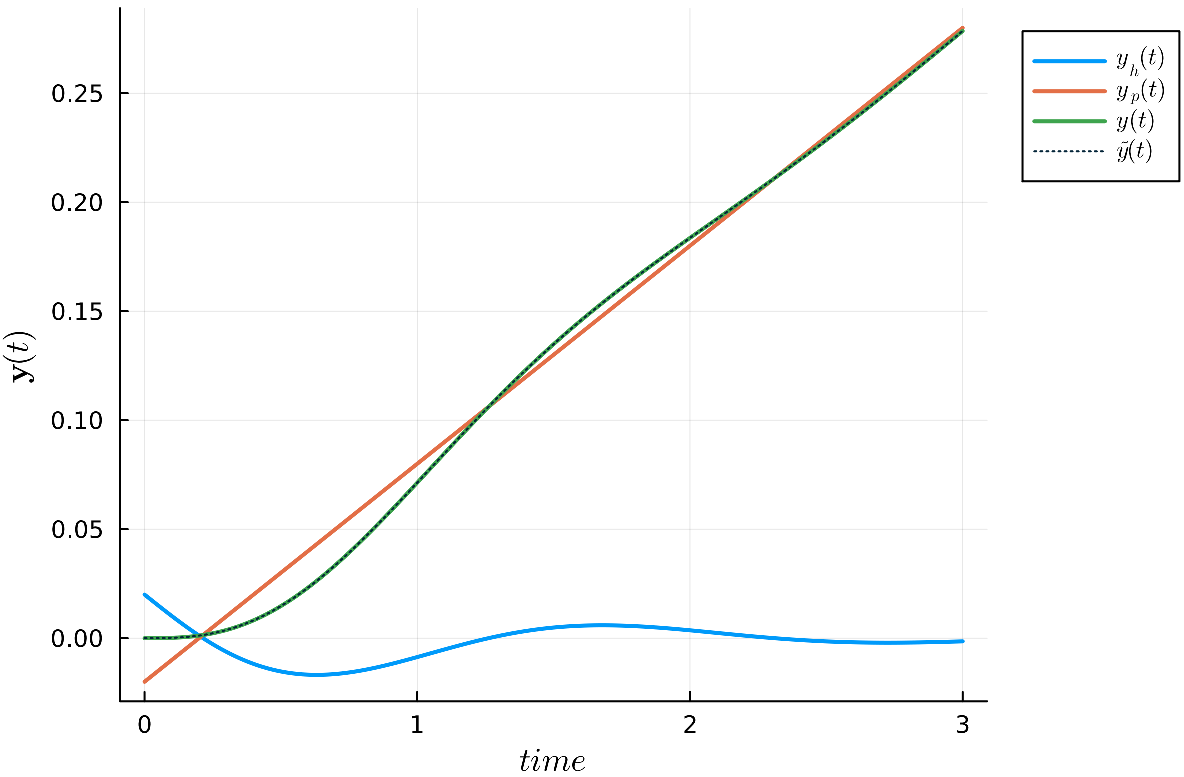

Figure 3 – Homogeneous solution $y_h(t)$ (solid blue line), permanent solution $y_p(t)$ (solid red line) and complete solution $y(t)$ (solid green line) obtained with the proposed approach for the polynomial excitation (linear ramp). The complete solution obtained with the Newmark-beta method $\tilde{y}(t)$ is shown as a dark dotted line.

2.5 SECOND ORDER ODES WITH CONSTANT COEFFICIENTS - DISCONTINUOUS EXCITATIONS

Dirac's delta excitation (or unitary impulse) can be used, for example, to model impact loading in mechanical systems and it is fundamental for real applications in machinery and infrastructure submitted to such type of loads, like hammers, pistons, shock absorbers, suspensions and landing gears, to name a few. Its a not trivial loading to model since it is not even a function. Thus, its modelling in numerical models is usually approximated to equivalent continuous functions with unitary integral withing a very small time span around the time of impact (EFTEKHARI, 2015).

Other important discontinuous excitation is the unitary step (Heaviside). Step functions can be used to parameterize various discontinuous excitations in engineering, especially in mechanical and electrical engineering, since mechanical force and electrical current can be switch on and off and have its amplitude modulated in a non-smooth way, (AGARWAL; LANG, 2005; IRWIN; NELMS, 2006; HUMAR, 2005; KELLY, 2006; KANWAL, 2011). In most applications, the solution of the differential equation with this type of excitation is given by Laplace Transform, (AGARWAL; LANG, 2005; IRWIN; NELMS, 2006; BOYCE; DIPRIMA, 2001; HUMAR, 2005), whose task of finding its inverse rapidly gains complexity as the excitation function becomes more complicated. Thus, the generalized integrating factor brings a systematic way of finding solutions for these important excitation class of functions without the need of resorting to inverse operations or inverse tables and partial fractions.

2.5.1 Particular solution due to step functions - Heaviside

Let the excitation be defined as a sum of functions $f_k(t)$ multiplied by a step functions,

$$\bar{f}(t) = \sum_{k=1}^{n_k} f_k(t)\mathscr{H}(t-t_k), \tag{119}$$

where

$$\mathscr{H}(t-t_k) = \begin{cases} 0 & t < t_k \\ 1 & t \geq t_k \end{cases} \tag{120}$$

is the Heaviside, or unitary step at t_k. Inserting Eq. (119) into Eq. (81) yields

$$y_p(t) = \exp\left(-f_{2,1,2}t\right)\int_0^t \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\int_0^t \exp\left(\hat{k}t\right)\sum_{k=1}^{n_k} f_k(t)\mathscr{H}(t-t_k)\,\mathrm{d}t\,\mathrm{d}t, \tag{121}$$

as integration is a linear operator,

$$y_p(t)=\sum_{k=1}^{n_k}\exp\left(-f_{2,1,2}t\right)\int_0^t \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\int_0^t \exp\left(\hat{k}t\right)f_k(t)\mathscr{H}(t-t_k)\,\mathrm{d}t\,\mathrm{d}t. \tag{122}$$

Using Eq. (360) in Eq. (122) for the inner integral results in

$$y_p(t)=\sum_{k=1}^{n_k}\exp\left(-f_{2,1,2}t\right)\int_0^t \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\int_{t_k}^t \exp\left(\hat{k}t\right)f_k(t)\,\mathrm{d}t\,\mathscr{H}(t-t_k)\,\mathrm{d}t, \tag{123}$$

and for the outer integral

$$y_p(t)=\sum_{k=1}^{n_k}\exp\left(-f_{2,1,2}t\right)\int_{t_k}^t \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\int_{t_k}^t \exp\left(\hat{k}t\right)f_k(t)\,\mathrm{d}t\,\mathrm{d}t\,\mathscr{H}(t-t_k). \tag{124}$$

2.5.1.1 Initial conditions and the Heaviside

Let a second order ordinary differential equation have a function $\bar{f}(t)$ multiplied by a Heaviside step as excitation,

$$\ddot{y}(t)+\bar{c}\dot{y}(t)+\bar{k}y(t)=\bar{f}(t)\mathscr{H}\left(t-t_H\right), \tag{125}$$

which is the same as solving two differential equations,

$$\begin{cases}\ddot{y}_1(t)+\bar{c}\dot{y}_1(t)+\bar{k}y_1(t)=0, & \text{if } t<t_H\\ \ddot{y}_2(t)+\bar{c}\dot{y}_2(t)+\bar{k}y_2(t)=\bar{f}(t). & \text{otherwise}\end{cases}, \tag{126}$$

At t_H, assuming continuity, the initial conditions of y_2 must be equal to the values of y_1 and its derivative at this point, *i.e.*, $y_1\left(t_H\right)=y_2\left(t_H\right)$ and $\dot{y}_1\left(t_H\right)=\dot{y}_2\left(t_H\right)$. As there is no excitation before t_H and a purely particular solution is sought after, the solution between $t=0$ and $t=t_H$ is $y_1(t)=0$ and, consequently $\dot{y}_1(t)=0$. Thus, it follows that $y_2(t_H)=0$ and $\dot{y}_2(t_H)=0$.

It is straightforward that this holds true even when $t_H\to 0$. Therefore, all solutions given by Eq. (124), *i.e.* using Heaviside as excitation, have $y_p(t)=0$ and $\dot{y}_p(t)=0$ as fixed initial conditions. Thus, the imposition of non-homogeneous initial conditions, $y\left(t_0\right)=u_0$ and $\dot{y}\left(t_0\right)=v_0$, at $t_0\leq t_H$ gets even simpler, through the following system of linear equations if $f_{2,1,2}-\hat{k}\neq 0$,

$$\begin{bmatrix}\exp\left(-f_{2,1,2}t_0\right) & \exp\left(-\hat{k}t_0\right)\\ -f_{2,1,2}\exp\left(-f_{2,1,2}t_0\right) & -\hat{k}\exp\left(-\hat{k}t_0\right)\end{bmatrix}\begin{Bmatrix}C_1\\ C_2\end{Bmatrix}=\begin{Bmatrix}u_0\\ v_0\end{Bmatrix}, \tag{127}$$

which, particularized for $t_0=0$, simplifies to

$$\begin{bmatrix} 1 & 1 \\ -f_{2,1,2} & -\hat{k} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix}, \tag{128}$$

whose solution is

$$C_2 = \frac{v_0 + f_{2,1,2} u_0}{f_{2,1,2} - \hat{k}}, \tag{129}$$

$$C_1 = u_0 - C_2. \tag{130}$$

Otherwise, if $f_{2,1,2} - \hat{k} = 0$, one must use Eq. (83) to assemble the system of equations,

$$\begin{bmatrix} \exp(-f_{2,1,2} t_0) & t_0 \exp(-f_{2,1,2} t_0) \\ -f_{2,1,2} \exp(-f_{2,1,2} t_0) & \exp(-f_{2,1,2} t_0) - t_0 f_{2,1,2} \exp(-f_{2,1,2} t_0) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix}, \tag{131}$$

which can be multiplied by $\exp(f_{2,1,2} t_0)$,

$$\begin{bmatrix} 1 & t_0 \\ -f_{2,1,2} & 1 - t_0 f_{2,1,2} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \exp(f_{2,1,2} t_0) \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix}, \tag{132}$$

and, then pivoted to

$$\begin{bmatrix} 1 & t_0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \exp(f_{2,1,2} t_0) \begin{Bmatrix} u_0 \\ v_0 + f_{2,1,2} u_0 \end{Bmatrix}. \tag{133}$$

If $t_0 = 0$, it is further simplified to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 + f_{2,1,2} u_0 \end{Bmatrix}, \tag{134}$$

whose solution is

$$C_2 = v_0 + f_{2,1,2} u_0, \tag{135}$$

$$C_1 = u_0. \tag{136}$$

Thus, the constants C_1 and C_2 can be evaluated without any knowledge about the derivative of the particular response.

Example with Heaviside function

Consider the problem from (BOYCE; DIPRIMA, 2001)

$$\ddot{y}(t)+4y(t)=\bar{f}(t)=\begin{cases}0, & 0\leq t<5\\ \frac{t-5}{5}, & 5\leq t<10\\ 1, & t\geq 10\end{cases}$$

with homogeneous initial conditions $u_0=v_0=0$ at $t_0=0$. The excitation function can be written as

$$\bar{f}(t)=\frac{t-5}{5}\mathscr{H}(t-5)-\frac{t-10}{5}\mathscr{H}(t-10). \tag{137}$$

Using Eq. (74)

$$f_{2,1,2}=\frac{0+\sqrt{0-16}}{2}=2i, \tag{138}$$

and using Eq. (124)

$$y_p(t)=e^{-2it}\left(\int_5^t e^{4it}\int_5^t e^{-2it}\left(\frac{t-5}{5}\right)\mathrm{d}t\,\mathrm{d}t\mathscr{H}(t-5)+\right.$$
$$\left.\int_{10}^t e^{4it}\int_{10}^t e^{-2it}\left(\frac{10-t}{5}\right)\mathrm{d}t\,\mathrm{d}t\mathscr{H}(t-10)\right) \tag{139}$$

such that

$$y_p(t)=\frac{\left(ie^{4it}+\left(4e^{10i}t-20e^{10i}\right)e^{2it}-ie^{20i}\right)e^{-2it-10i}}{80}\mathscr{H}(t-5)+$$
$$-\frac{\left(ie^{4it}+\left(4e^{20i}t-40e^{20i}\right)e^{2it}-ie^{40i}\right)e^{-2it-20i}}{80}\mathscr{H}(t-10). \tag{140}$$

The complete analytical solution given in (BOYCE; DIPRIMA, 2001) (using Laplace Transform) is

$$y(t)=\left[\frac{t-5}{4}-\frac{sin(2t-10)}{8}\right]\frac{\mathscr{H}(t-5)}{5}-\left[\frac{t-10}{4}-\frac{sin(2t-20)}{8}\right]\frac{\mathscr{H}(t-10)}{5}, \tag{141}$$

the same result of Eq. (140) after using the Euler identity (as the homogeneous response is zero in this example).

Figure 4 shows the (real part of) the homogeneous solution (solid blue line), the permanent solution, Eq. (140), (solid red line) and the complete response (solid green line). As the initial conditions are null, the homogeneous solution is always zero and the green line overlaps the red line for all t. The reference solution, Eq. (141) is shown as a dark dotted line. The maximum complex part of the complete solution has a magnitude of 1×10^{-18}, negligible when compared to the real part.

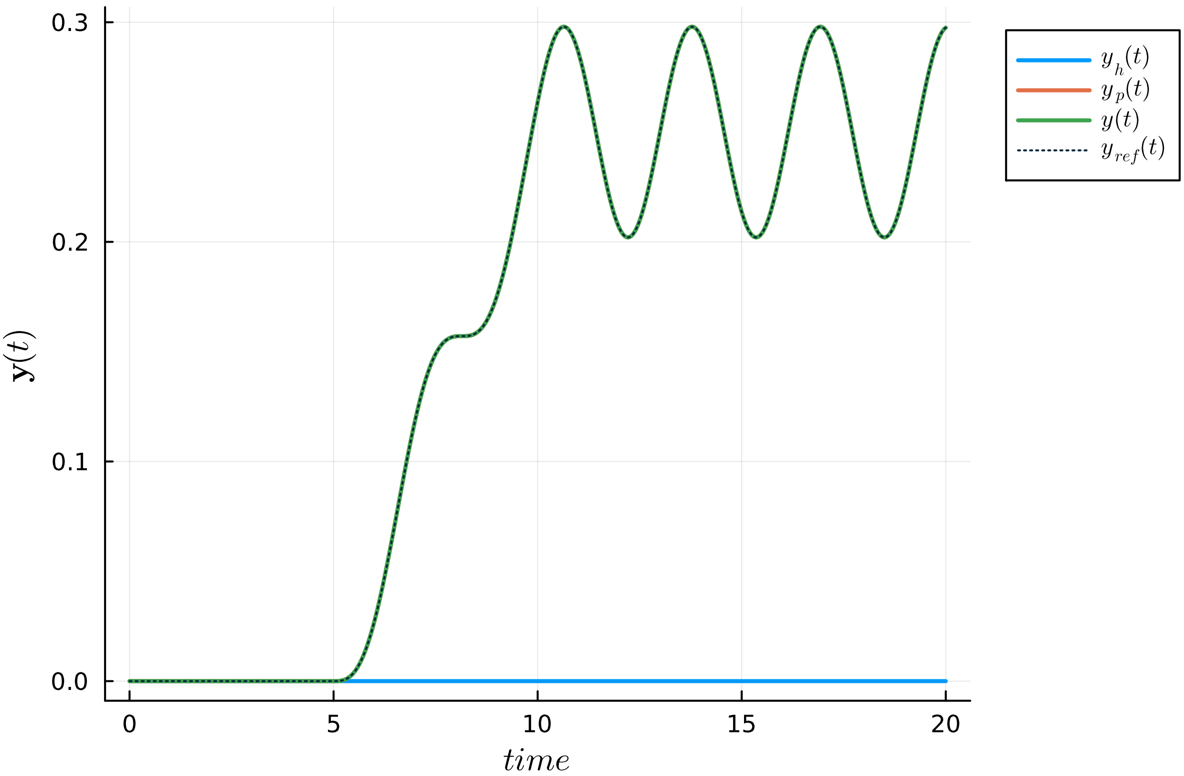

Figure 4 – Homogeneous solution (solid blue line), particular solution (solid red line) and complete response (solid green line) due to an unitary impulse at $t = 5$s. Reference solution given by (BOYCE; DIPRIMA, 2001) is shown as a dark dotted line.

2.5.2 Particularizing $f_k(t)$ as a polynomial

Analytical functions can be approximated using polynomials, the intuition behind Taylor series and polynomial regression. Thus, polynomial functions play an important role in analysis, applied mathematics and engineering, which makes them an interesting particularization for $f_k(t)$ in Eq. (119). Let the excitation function $\bar{f}(t)$ be given by

$$\bar{f}(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} c_{k,l} t^l \mathscr{H}(t - t_k), \tag{142}$$

a series of polynomials of order n_{lk} multiplied by Heavisides at times t_k. As an example, consider the loading shown in Figure 5. This complicate loading can be written using Eq. (142) as

$$\bar{f}(t) = (2t)\mathscr{H}(t-0) + \left(-1+2t-t^2\right)\mathscr{H}(t-1) + \left(3-4t+t^2\right)\mathscr{H}(t-3) + (-2)\mathscr{H}(t-5), \tag{143}$$

such that $n_k = 4$.

The particular solution for such general loading can be obtained in closed form. Applying Eq. (142) in Eq. (124) and using the linearity of the integration operator yields

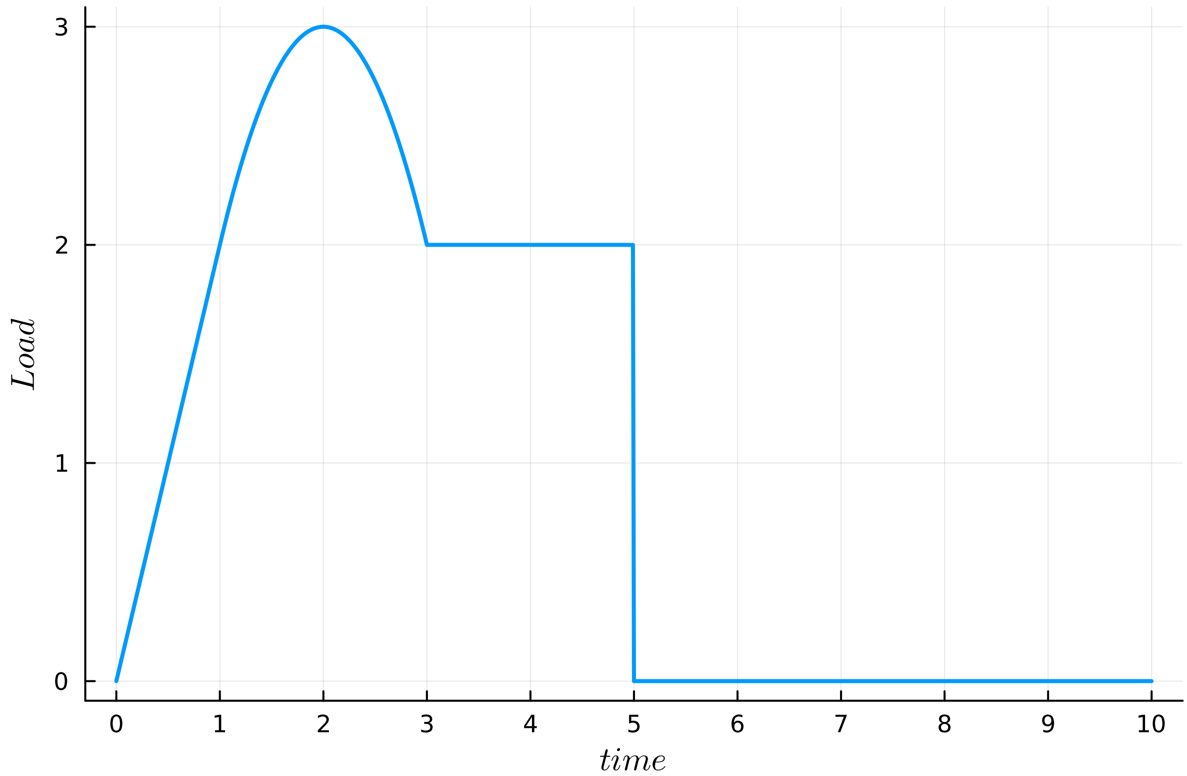

Figure 5 – Loading comprised of a linear ramp $(0 < t < 1)$, a quadratic equation $(1 < t < 3)$ a constant value $(3 < t < 5)$ and a sudden change to 0 for $t > 5$.

$$y_p(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} \exp\left(-f_{2,1,2}t\right) \int_{t_k}^{t} \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \int_{t_k}^{t} \exp\left(\hat{k}t\right) c_{k,l} t^l \, \mathrm{d}t \, \mathrm{d}t \, \mathscr{H}(t - t_k). \quad (144)$$

The inner convolution is evaluated by using Eq. (108)

$$y_p(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} c_{k,l} \exp\left(-f_{2,1,2}t\right) \int_{t_k}^{t} \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t^s \exp\left(\hat{k}t\right) - \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \exp\left(\hat{k}t_k\right) \right) \mathrm{d}t \, \mathscr{H}(t - t_k), \quad (145)$$

where $s = l - p + 1$. To further evaluate this integral, two cases will be separately addressed - $f_{2,1,2} - \hat{k} \neq 0$ and the opposite, respectively. Thus, for the first case, the integral of the outer convolution is split among its integrands,

$$y_p(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} c_{k,l} \exp\left(-f_{2,1,2}t\right) \left(\int_{t_k}^{t} \exp\left(f_{2,1,2}t\right) \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t^s \, \mathrm{d}t - \int_{t_k}^{t} \exp\left(f_{2,1,2}t + \hat{k}\left(t_k - t\right)\right) \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \, \mathrm{d}t \right) \mathscr{H}(t - t_k). \quad (146)$$

Taking the constant terms out of the integrals

$$y_p(t) = \sum_{k=1}^{n_k}\sum_{l=0}^{n_{lk}} c_{k,l}\exp\left(-f_{2,1,2}t\right)\left(\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}\int_{t_k}^{t}\exp\left(f_{2,1,2}t\right)t^s\,\mathrm{d}t\right.$$
$$\left.-\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t_k^s\int_{t_k}^{t}\exp\left(f_{2,1,2}t+\hat{k}\left(t_k-t\right)\right)\mathrm{d}t\right)\mathscr{H}(t-t_k), \tag{147}$$

which has, then, the first convolution evaluated again using Eq. (108), and the second convolution evaluated by simple integration,

$$y_p(t) = \sum_{k=1}^{n_k}\mathscr{H}(t-t_k)\sum_{l=0}^{n_{lk}} c_{k,l}\exp\left(-f_{2,1,2}t\right)\left\{\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}\left\{\sum_{q=1}^{s+1}(-1)^{q+1}\right.\right.$$
$$\left(\frac{1}{f_{2,1,2}}\right)^q\frac{(s)!}{(s+1-q)!}t^{s+1-q}\exp\left(f_{2,1,2}t\right)-\sum_{q=1}^{s+1}(-1)^{q+1}\left(\frac{1}{f_{2,1,2}}\right)^q$$
$$\left.\frac{s!}{(s+1-q)!}t_k^{s+1-q}\exp\left(f_{2,1,2}t_k\right)\right\}-\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t_k^s$$
$$\left.\left\{\exp\left(f_{2,1,2}t+\hat{k}\left(t_k-t\right)\right)-\exp\left(f_{2,1,2}t_k+\hat{k}\underbrace{\left(t_k-t_k\right)}_{0}\right)\right\}\frac{1}{f_{2,1,2}-\hat{k}}\right\}. \tag{148}$$

Rearranging the terms

$$y_p(t) = \sum_{k=1}^{n_k}\mathscr{H}(t-t_k)\sum_{l=0}^{n_{lk}} c_{k,l}\left\{\left\{\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}\sum_{q=1}^{s+1}(-1)^{q+1}\left(\frac{1}{f_{2,1,2}}\right)^q\right.\right.$$
$$\left.\frac{s!}{(s+1-q)!}t^{s+1-q}\right\}+\exp\left(f_{2,1,2}\left(t_k-t\right)\right)\left\{\frac{1}{f_{2,1,2}-\hat{k}}\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t_k^s\right.$$
$$\left.-\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}\sum_{q=1}^{s+1}(-1)^{q+1}\left(\frac{1}{f_{2,1,2}}\right)^q\frac{s!}{(s+1-q)!}t_k^{s+1-q}\right\}$$
$$\left.-\exp\left(\hat{k}\left(t_k-t\right)\right)\left(\frac{1}{f_{2,1,2}-\hat{k}}\right)\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t_k^s\right\}, \tag{149}$$

which, with further simplification, can be written as

$$y_p(t)=\sum_{k=1}^{n_k}\mathscr{H}(t-t_k)\sum_{l=0}^{n_{lk}}c_{k,l}\left\{\left\{\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}\sum_{q=1}^{s+1}(-1)^{q+1}\left(\frac{1}{f_{2,1,2}}\right)^{q}\right.\right.$$
$$\left.\frac{s!}{(s+1-q)!}t^{s+1-q}\right\}+\exp\left(f_{2,1,2}\left(t_k-t\right)\right)\left\{\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}\left\{\frac{t_k^s}{f_{2,1,2}-\hat{k}}\right.\right.$$
$$\left.\left.-\sum_{q=1}^{s+1}(-1)^{q+1}\left(\frac{1}{f_{2,1,2}}\right)^{q}\frac{s!}{(s+1-q)!}t_k^{s+1-q}\right\}\right\}$$
$$\left.-\exp\left(\hat{k}\left(t_k-t\right)\right)\left(\frac{1}{f_{2,1,2}-\hat{k}}\right)\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t_k^s\right\}. \tag{150}$$

Now, for the second case, *i.e.* $f_{2,1,2}-\hat{k}=0$, Equation (145) can have the integrand split in two parcels, one that is dependent upon t and, hence, can be integrated just like in the first case, and the one parcel that is constant, to which the condition $f_{2,1,2}-\hat{k}=0$ must be applied,

$$y_p(t)=\sum_{k=1}^{n_k}\sum_{l=0}^{n_{lk}}c_{k,l}\exp\left(-f_{2,1,2}t\right)\left[\int_{t_k}^{t}\exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t^s\exp\left(\hat{k}t\right)\mathrm{d}t\right.$$
$$\left.-\int_{t_k}^{t}\exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t_k^s\exp\left(\hat{k}t_k\right)\mathrm{d}t\right]\mathscr{H}\left(t-t_k\right). \tag{151}$$

Applying $f_{2,1,2}-\hat{k}=0$ and that the integral is a linear operator, terms can be rearranged to

$$y_p(t)=\sum_{k=1}^{n_k}\sum_{l=0}^{n_{lk}}c_{k,l}\exp\left(-f_{2,1,2}t\right)\left[\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}\int_{t_k}^{t}\exp\left(f_{2,1,2}t\right)t^s\,\mathrm{d}t\right.$$
$$\left.-\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t_k^s\exp\left(\hat{k}t_k\right)\int_{t_k}^{t}1\,\mathrm{d}t\right]\mathscr{H}\left(t-t_k\right). \tag{152}$$

The first convolution in previous equation can be evaluated using Eq. (108),

$$y_p(t)=\sum_{k=1}^{n_k}\sum_{l=0}^{n_{lk}}c_{k,l}\exp\left(-f_{2,1,2}t\right)\left[\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}\left[\sum_{q=1}^{s+1}(-1)^{q+1}\left(\frac{1}{f_{2,1,2}}\right)^{q}\right.\right.$$
$$\left(\frac{s!}{(s+1-q)!}t^{s+1-q}\exp\left(f_{2,1,2}t\right)\right)-\sum_{q=1}^{s+1}(-1)^{q+1}\left(\frac{1}{f_{2,1,2}}\right)^{q}\frac{s!}{(s+1-q)!}$$
$$\left.\left.t_k^{s+1-q}\exp\left(f_{2,1,2}t_k\right)\right]-\sum_{p=1}^{l+1}(-1)^{p+1}\left(\hat{k}\right)^{-p}\frac{l!}{s!}t_k^s\exp\left(\hat{k}t_k\right)\left(t-t_k\right)\right]\mathscr{H}\left(t-t_k\right), \tag{153}$$

which can be finally simplified to

$$y_p(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} c_{k,l} \left[\sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} \left[\sum_{q=1}^{s+1} (-1)^{q+1} \left(\frac{1}{f_{2,1,2}} \right)^q \right.\right.$$

$$\left.\left. \frac{s!}{(s+1-q)!} \left(t^{s+1-q} - t_k^{s+1-q} \exp\left(f_{2,1,2}\left(t_k - t\right)\right) \right) \right] - \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} \right.$$

$$\left. t_k^s \exp\left(\hat{k} t_k - f_{2,1,2} t\right) (t - t_k) \right] \mathscr{H}\left(t - t_k\right). \qquad (154)$$

Thus, two expressions were derived for the particular response due to Heaviside steps multiplied by polynomials - one for critical damping, Eq. (154), and another for any other damping, Eq. (150).

Example

Consider a mechanical system described by

$$\ddot{y}(t) + \dot{y}(t) + 4y(t) = \bar{f}(t) \qquad (155)$$

with non-homogeneous initial conditions $u_0 = -0.2$ and $v_0 = 0.1$ at $t_0 = 0$. The loading is given by

$$\bar{f}(t) = (2t)\,\mathscr{H}(t-0) + \left(-1 + 2t - t^2\right)\mathscr{H}(t-1) + \left(3 - 4t + t^2\right)\mathscr{H}(t-3) + (-2)\,\mathscr{H}(t-5), \qquad (156)$$

shown in Figure 5.

Hence, $\bar{c}^2 - 4\bar{k} = -\frac{15}{4} \implies f_{2,1,2} - \hat{k} \neq 0$, and Equation (150) can be used. Figure 6 shows the solution obtained for this example. The homogeneous response is shown as a solid blue line, the particular response as a solid red line and the complete response as a solid green line. The numerical solution obtained using the Newmark-beta method with $\Delta t = 0.001$ s is shown as a dark dotted line.

2.5.3 Particular solution due to unitary impulses - Dirac's deltas

Consider that the normalized excitation $\bar{f}(t)$ is given by n_δ Dirac's deltas at times t_k

$$\bar{f}(t) = \sum_{k=1}^{n_\delta} c_k \delta(t - t_k), \qquad (157)$$

with coefficients $c_k \in \mathbb{R}$. Particular solution given by Eq. (81) can be written as

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int_0^t \exp\left(\hat{k}t\right) \sum_{k=1}^{n_\delta} c_k \delta(t - t_k)\, \mathrm{d}t \right) \mathrm{d}t, \qquad (158)$$

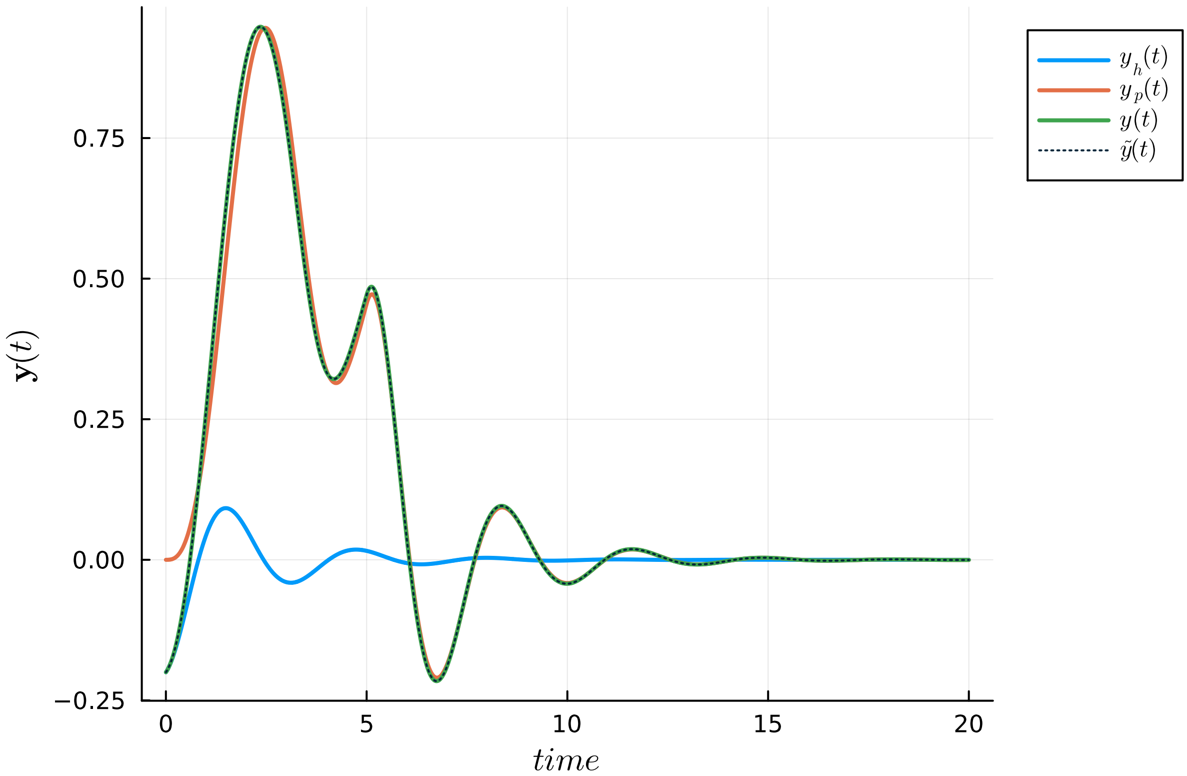

Figure 6 – Homogeneous response (solid blue line), particular response (solid red line) and complete response (solid green line) for the loading $\bar{f}(t)$ given in Fig. 5 and non-homogeneous initial conditions. The complete solution obtained using the Newmark-beta method is shown as a dark dotted line.

or, using the linearity of the integral

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right) \left(\int_0^t \exp\left(\hat{k}t\right) \delta\left(t - t_k\right) \mathrm{d}t\right) \mathrm{d}t. \tag{159}$$

The inner integral can be found by using the Filtering property of the Dirac's delta (Appendix A.3, Eq. (633)), such that

$$\int_0^t \exp\left(\hat{k}t\right) \delta\left(t - t_k\right) \mathrm{d}t = \exp\left(\hat{k}t_k\right) \mathscr{H}\left(t - t_k\right) \tag{160}$$

where $\mathscr{H}(t - t_k)$ is the Heaviside function at t_k. Thus,

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right) \exp\left(\hat{k}t_k\right) \mathscr{H}\left(t - t_k\right) \mathrm{d}t, \tag{161}$$

or

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp\left(-f_{2,1,2}t\right) \exp\left(\hat{k}t_k\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right) \mathscr{H}\left(t - t_k\right) \mathrm{d}t. \tag{162}$$

This convolution over a function multiplied by the Heaviside function can be evaluated splitting the integration domain according to the step function, thus, resulting in a change of integration limits,

$$\int_0^t \exp(\beta t)\, f_k(t)\mathscr{H}(t-t_k)\,\mathrm{d}t = \underbrace{\int_0^{t_k} \exp(\beta t)\, f_k(t)\mathscr{H}(t-t_k)\,\mathrm{d}t}_{=0} + \int_{t_k}^t \exp(\beta t)\, f_k(t)\mathscr{H}(t-t_k)\,\mathrm{d}t, \tag{163}$$

where β is a generic coefficient, such that

$$\int_0^t \exp(\beta t)\, f_k(t)\mathscr{H}(t-t_k)\,\mathrm{d}t = \left(\int_{t_k}^t \exp(\beta t)\, f_k(t)\,\mathrm{d}t\right)\mathscr{H}(t-t_k). \tag{164}$$

Thus, Eq. (162) can be written as

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp(-f_{2,1,2}t)\exp\left(\hat{k}t_k\right)\int_{t_k}^t \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\mathrm{d}t\;\mathscr{H}(t-t_k). \tag{165}$$

As in Eq. (145), Equation (165) also must be treated in two different cases. For the first one, when $f_{2,1,2}-\hat{k}=0$,, the particular solution is given by

$$y_p(t) = \sum_{k=1}^{n_\delta} \frac{c_k}{f_{2,1,2}-\hat{k}}\left(\exp\left(\hat{k}(t_k-t)\right) - \exp\left(f_{2,1,2}(t_k-t)\right)\right)\mathscr{H}(t-t_k). \tag{166}$$

For the opposite case, when damping is critical, the particular solution is given by

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp\left(\hat{k}t_k - f_{2,1,2}t\right)(t-t_k)\,\mathscr{H}(t-t_k). \tag{167}$$

It must be observed that the particular solution contains a Heaviside such that the results obtained in Appendix C.2 are also valid for unitary impulse as excitation.

Additionally, for under damped problems, $f_{2,1,2}$ is complex and $\hat{k} = \frac{\bar{k}}{f_{2,1,2}} = f^*_{2,1,2}$ (where $*$ stands for complex-conjugate). Thus

$$y_p(t) = \sum_{k=1}^{n_\delta} \frac{c_k}{2i\Im(f_{2,1,2})}\left(e^{f^*_{2,1,2}(t_k-t)} - e^{f_{2,1,2}(t_k-t)}\right)\mathscr{H}(t-t_k), \tag{168}$$

or

$$y_p(t) = \sum_{k=1}^{n_\delta} \frac{c_k}{2i\Im(f_{2,1,2})} e^{\Re(f_{2,1,2})(t_k-t)}\left(e^{-i\Im(f_{2,1,2})(t_k-t)} - e^{i\Im(f_{2,1,2})(t_k-t)}\right)\mathscr{H}(t-t_k) \tag{169}$$

and user Euler's identity for sin

$$y_p(t) = \sum_{k=1}^{n_\delta} \frac{c_k}{2\Im(f_{2,1,2})} e^{\Re(f_{2,1,2})(t_k-t)} \sin\left(\Im(f_{2,1,2})(t-t_k)\right)\mathscr{H}(t-t_k). \tag{170}$$

First example with unitary impulse

Consider a mechanical system described by the following ODE

$$2\ddot{y}(t)+\dot{y}(t)+2y(t)=\delta(t-5), \tag{171}$$

with homogeneous initial conditions $u_0=0$ and $v_0=0$ for $t_0=0$. The analytical solution is (BOYCE; DIPRIMA, 2001)

$$y_p(t)=\begin{cases} 0 & t<5 \\ \frac{2}{\sqrt{15}}e^{\frac{5-t}{4}}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right) & t\geq 5 \end{cases}. \tag{172}$$

From the data, $\bar{c}=1/2$, $\bar{k}=1$, $n_\delta=1$ and $t_1=5$. Hence, $\bar{c}^2-4\bar{k}=-\frac{15}{4} \implies f_{2,1,2}-\hat{k}\neq 0$. Constant $f_{2,1,2}$ can be found by solving Eq. (74) such that $f_{2,1,2}=0.25+0.96824i$. Equation (166) reduces to

$$y_p(t)=\frac{1}{2(0.96824)i}\left(e^{(0.25-0.9682i)(5-t)}-e^{(0.25+0.96824i)(5-t)}\right)\mathscr{H}(t-5) \tag{173}$$

and, although complex, has negligible complex values, matching the analytical solution. Alternatively, using Eq. (170)

$$y_p(t)=\frac{1}{2(0.96824)}e^{0.25(5-t)}\sin\left(0.96824(t-5)\right)\mathscr{H}(t-5) \tag{174}$$

which is also identical to the analytical solution provided by (BOYCE; DIPRIMA, 2001). Figure 7 shows that the homogeneous solution (solid blue line) is always zero, as expected (due to the homogeneous initial conditions of this example). The complete solution (solid green line) obtained with the proposed approach overlaps the solid red line of the particular response, since they are always equal in this example. The analytical reference solution, (BOYCE; DIPRIMA, 2001), is shown as a dark dotted line.

Second example with unitary impulse

Consider a mechanical system subjected to two opposite impacts at $t=1$ and $t=5$s

$$\ddot{y}(t)+\dot{y}(t)+4y(t)=\delta(t-1)-\delta(t-5), \tag{175}$$

with non-homogeneous initial conditions $u_0=-0.2$ and $v_0=0.1$ for $t_0=0$.

From the data, $\bar{c}=1$, $\bar{k}=4$ and $n_\delta=2$, with $t_1=1$, $c_1=1.0$, $t_2=5$ and $c_2=-1$. Hence, $\bar{c}^2-4\bar{k}=-15 \implies f_{2,1,2}-\hat{k}\neq 0$. Constant $f_{2,1,2}$ can be found by solving Eq. (74) such that $f_{2,1,2}=0.5+1.93649i$, an under damped problem.

Figure 8 shows the (real part of) homogeneous solution $y_h(t)$ (solid blue line), the permanent solution $y_p(t)$ (solid red line) and the complete solution $y(t)$ (solid green line) obtained with the proposed approach for the two opposite unitary impulses at $t=1$ and $t=5$s. It is worth noticing that the complete solution is equal to $y_h(t)$ for $t<1$ s (before the first unitary

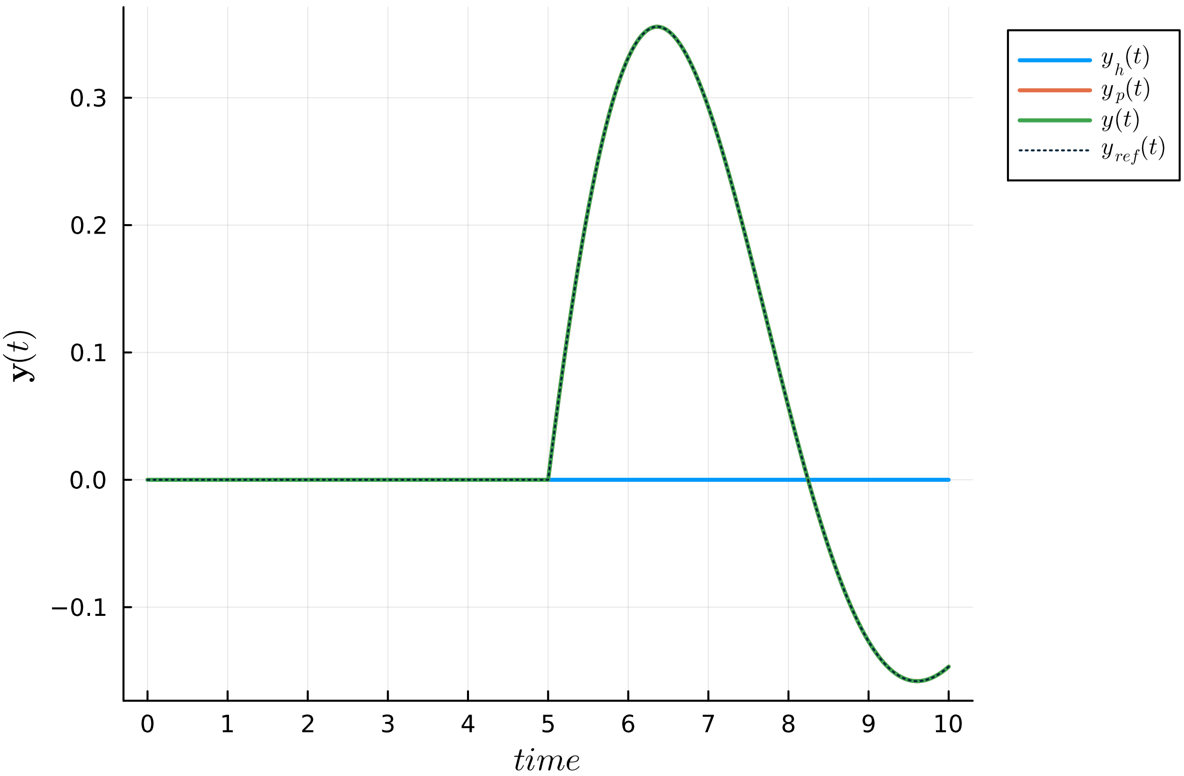

Figure 7 – Homogeneous solution $y_h(t)$ (solid blue line), permanent solution $y_p(t)$ (solid red line) and complete solution $y(t)$ (solid green line) obtained with the proposed approach for unitary impulse at 5s. The analytical reference solution $y_{ref}(t)$ is shown as a dotted line. The red line cannot be seen in this image since the green line (complete solution) is over the red line (particular solution) for all t.

impulse), as expected. The complex part of the solution is negligible when compared to the real part (numerically zero) for all t.

Regarding the numerical solution, each impulse was approximated by (EFTEKHARI, 2015)

$$\delta(t-t_0) \approx \frac{1}{2\varepsilon}\left(1+\cos\left(\frac{\pi(t-t_0)}{\varepsilon}\right)\right) \quad t_0-\varepsilon \leq t \leq t_0+\varepsilon. \tag{176}$$

with $\varepsilon = \Delta t$ (the proposed methodology does not need such approximation). The complete solution obtained with the Newmark-beta method $\tilde{y}(t)$ (dotted line) matches the complete solutions obtained with the proposed approach when a small time step of $\Delta t = 0.001$s is used.

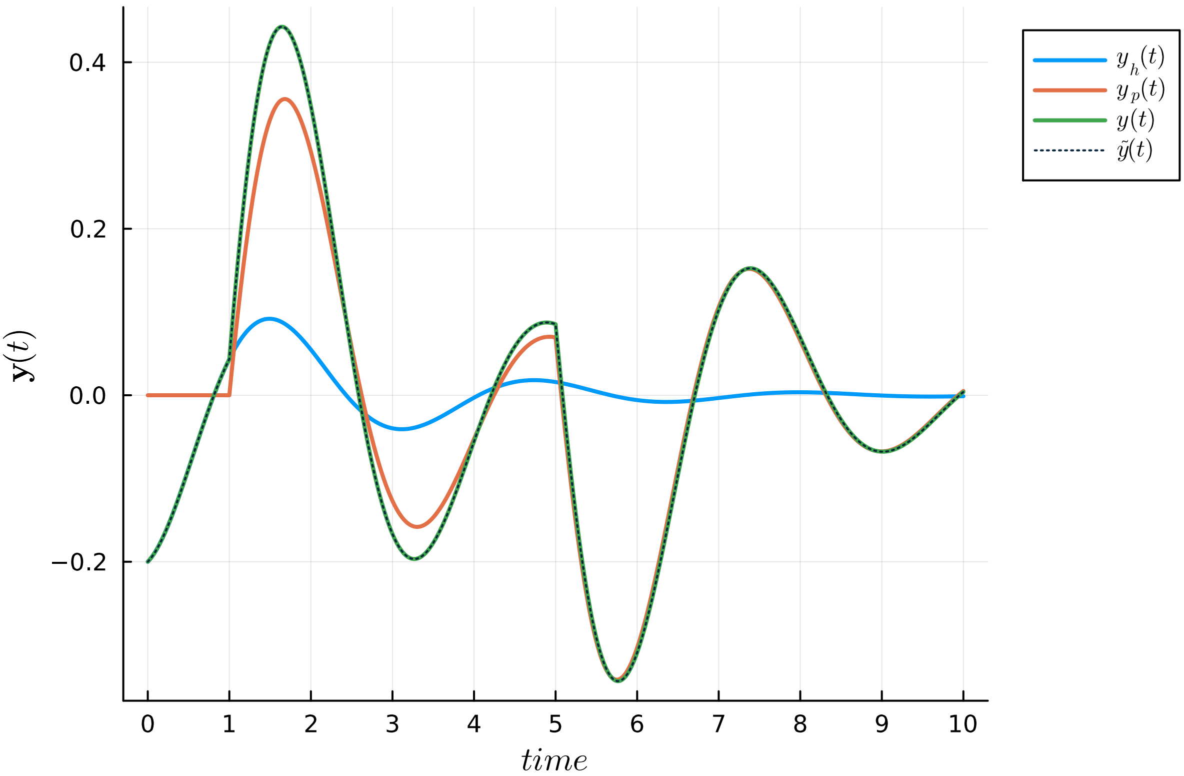

Figure 8 – Homogeneous solution $y_h(t)$ (solid blue line), permanent solution $y_p(t)$ (solid red line) and complete solution $y(t)$ (solid green line) obtained with the proposed approach for two opposite unitary impulses at $t = 1$ and $t = 5$. The complete solution obtained with the Newmark-beta method $\tilde{y}(t)$ is shown as a dark dotted line.

2.6 FINAL REMARKS OF THE CHAPTER

This chapter presented the formulation for the generalization of the Leibniz integrating factor to linear ODEs with order $m \geq 2$. The new method, Generalized integrating Factor or GIF, is a systematic approach to reduce order of the original ODE by applying integrating factors dependent upon the independent variable only. The technique was applied to important ODEs and the case of constant coefficients was thoroughly studied due to its application in Engineering.

Various excitation functions yielded analytical solutions in closed form, like periodic and polynomial functions. Closed form solutions were also obtained for discontinuous excitation, as for Dirac's delta impulse and Heaviside step function. The Heaviside case was particularized for steps multiplied by polynomials, enlarging the range of applications with analytical solutions.

In contrast to previous and well established methods, like undetermined coefficients, variation of parameters and Laplace transform, no knowledge of a candidate solution, nor of the homogeneous solution, nor of the calculation of a inverse transformation was needed. The solution was instead analytically derived by means of double convolutions. Table 1 shows a summary of the main drawbacks of some well-established methods when compared to the proposed approach. It is worth noting that, unlike most of the traditional approaches, the proposed method can be used for time-dependent coefficients. As it will be discussed in the next chapter, these drawbacks make it hard to use most well-established methods to solve coupled systems of ODEs, unlike the proposed approach.

Table 1 – Main drawbacks of well-established methods when compared to the proposed approach

Method	Drawback when compared to the proposed approach
Integral transforms [1]	Need complicated algebraic operations and inverse transforms
State variables	Doubles the dimensionality of the problem
Undetermined coefficients	Needs a candidate particular solution
Variation of parameters	Needs to know the homogeneous solution beforehand
Characteristic polynomial	Needs a candidate homogeneous solution

[1] An example of integral transform is the Laplace transform.

Other interesting characteristic of the proposed approach is the fact that the solution procedure does not make any assumption about the level of damping (coefficient c) and can be used to sub, critically or super dampened problems without modifications.

For differential equations with general coefficients, the integrating factor depends on a particular solution of a Riccati differential equation. It was shown that the coefficients themselves might help finding a particular solution rather easily. Thus, the Riccati differential equation poses no strictly direct barrier to the wide application of the method, capable of giving accurate results and requiring no assumptions of solution candidates. Nonetheless, since this chapter focuses on the constant coefficient case, a more profound study of the solution of the Riccati equation is not carried out and is left for future works.

3 EXTENSION OF THE GENERALIZED INTEGRATING FACTOR TO SYSTEMS OF LINEAR ODES

Following from the applications in vibration analysis and electric circuitry simulation, which were discussed in Chapter 2, it is clear that solving systems of coupled linear ODEs is paramount. One such example is vibrations in discrete mechanical systems, like in automobile suspension systems, and another is vibrations in continuum systems discretized by FEM or BEM (Boundary Element Method).

Given the importance of systems of linear ODEs in Engineering, this chapter will extend the Generalized Integrating Factor to these systems. Also stemming from the applications, the formulation will be particularized for constant matrix coefficients. The homogeneous and the particular solutions will also be given in separate. The particular solution, for instance, will be studied for different kinds of excitation, both continuous and discontinuous.

It will be shown that the solution of the system of ODEs with constant matrix coefficients is linked to the solution of a matrix quadratic equation, whose solution is affected by damping. In particular, it will be shown that the matrix quadratic equation has closed-form solution when proportional damping is used, *i.e.* when the system has *classical normal modes*.

Running time experiments were carried out to assess the computational effort required by the technique to solve such systems of ODEs. Then, the Generalized Integrating Factor was compared to the Newmark-beta method and to the State Variables method. It was observed that the proposed technique delivered exact solutions in the smallest time when compared to those methods. Thus, the GIF method presents a clear advantage in both accuracy and in computational effort.

3.1 THE GENERALIZED INTEGRATING FACTOR FOR SECOND ORDER COUPLED SYSTEMS OF ODES

Consider the coupled system of n second order ODEs presented in Eq. (1), where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are time-dependent $n \times n$ matrices, $\mathbf{f}$ is a general vector also depending on time t and vector $\mathbf{y}$ is the solution. For easy of notation, explicit dependency on time t, (t), will not be carried out in the following equations. Also, $\mathbf{M}$ is invertible and, consequently, Eq. (1) can be multiplied by $\mathbf{M}^{-1}$ to the left, such that

$$\mathbf{I}\ddot{\mathbf{y}} + \bar{\mathbf{C}}\dot{\mathbf{y}} + \bar{\mathbf{K}}\mathbf{y} = \bar{\mathbf{f}}. \tag{177}$$

To solve this equation one start by splitting $\bar{\mathbf{C}}$ as

$$\bar{\mathbf{C}} = \mathbf{F}_{2,1,1} + \mathbf{F}_{2,1,2} \tag{178}$$

where both $\mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,2}$ are also $n \times n$ time dependent matrices, not necessarily real. Thus,

$$\underbrace{\mathbf{I}\ddot{\mathbf{y}} + \mathbf{F}_{2,1,1}\dot{\mathbf{y}}}_{\pi_{2,2}} + \underbrace{\mathbf{F}_{2,1,2}\dot{\mathbf{y}} + \bar{\mathbf{K}}\mathbf{y}}_{\pi_{2,1}} = \bar{\mathbf{f}}. \tag{179}$$

Next step is to multiply Eq. (179) by an invertible $n \times n$ matrix $\boldsymbol{\mu}_2$ (generalized integrating factor) such that

$$\underbrace{\boldsymbol{\mu}_2\mathbf{I}}_{\mathbf{P}_{2,2}}\,\ddot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,1}}_{\dot{\mathbf{P}}_{2,2}}\,\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,2}}_{\mathbf{P}_{2,1}}\,\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\bar{\mathbf{K}}}_{\dot{\mathbf{P}}_{2,1}}\,\mathbf{y} = \boldsymbol{\mu}_2\bar{\mathbf{f}}. \tag{180}$$

Using the definition of $\mathbf{P}_{2,2}$ and its time derivative

$$\dot{\mathbf{P}}_{2,2} = \boldsymbol{\mu}_2\mathbf{F}_{2,1,1} = (\dot{\overline{\boldsymbol{\mu}_2\mathbf{I}}}) = \dot{\boldsymbol{\mu}}_2 \tag{181}$$

and multiplying by $\boldsymbol{\mu}_2^{-1}$ results in

$$\boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2 = \mathbf{F}_{2,1,1} \tag{182}$$

such that it is fair to assume, when $\mathbf{F}_{2,1,1}$ and its integral commute,

$$\boldsymbol{\mu}_2 = \exp\left(\int \mathbf{F}_{2,1,1}\,\mathrm{d}t\right). \tag{183}$$

The same procedure can be applied to $\mathbf{P}_{2,1}$ and its time derivative

$$\dot{\mathbf{P}}_{2,1} = \boldsymbol{\mu}_2\bar{\mathbf{K}} = (\dot{\overline{\boldsymbol{\mu}_2\mathbf{F}_{2,1,2}}}) = \dot{\boldsymbol{\mu}}_2\mathbf{F}_{2,1,2} + \boldsymbol{\mu}_2\dot{\mathbf{F}}_{2,1,2} \tag{184}$$

and multiplying by $\boldsymbol{\mu}_2^{-1}$ to the left

$$\bar{\mathbf{K}} = \boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2\mathbf{F}_{2,1,2} + \dot{\mathbf{F}}_{2,1,2} \tag{185}$$

such that

$$\boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2 = \left(\bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,2}\right)\mathbf{F}_{2,1,2}^{-1} \tag{186}$$

with solution

$$\boldsymbol{\mu}_2 = \exp\left(\int \left(\bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,2}\right)\mathbf{F}_{2,1,2}^{-1}\,\mathrm{d}t\right). \tag{187}$$

Equating Eqs. (182) and (186)

$$\mathbf{F}_{2,1,1} = \left(\bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,2}\right)\mathbf{F}_{2,1,2}^{-1} \tag{188}$$

such that

$$\mathbf{F}_{2,1,1}\mathbf{F}_{2,1,2} = \left(\bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,2}\right) \tag{189}$$

and using Eq. (178)

$$\mathbf{F}_{2,1,1}\left(\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right) = \left(\bar{\mathbf{K}} - \dot{\bar{\mathbf{C}}} + \dot{\mathbf{F}}_{2,1,1}\right) \tag{190}$$

such that

$$\mathbf{F}_{2,1,1}^2 = \dot{\bar{\mathbf{C}}} + \mathbf{F}_{2,1,1}\bar{\mathbf{C}} - \bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,1} \tag{191}$$

is a coupled system of Riccati differential equations. Thus, by solving Eq. (191) it is possible to find $\boldsymbol{\mu}_2$ using Eq. (183). As both $\mathbf{P}_{2,2}$ and $\mathbf{P}_{2,1}$ depend on known $\boldsymbol{\mu}_2$, it is possible to re-write Eq. (180) as

$$(\dot{\mathbf{P}_{2,2}\dot{\mathbf{y}}}) + (\dot{\mathbf{P}_{2,1}\mathbf{y}}) = \boldsymbol{\mu}_2\bar{\mathbf{f}} \tag{192}$$

and integrating w.r.t time

$$(\mathbf{P}_{2,2}\dot{\mathbf{y}}) + (\mathbf{P}_{2,1}\mathbf{y}) = \int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \mathbf{C}_2 \tag{193}$$

where $\mathbf{C}_2$ is a constant vector. Multiplying by another invertible time dependent matrix $\boldsymbol{\mu}_1$ (generalized integrating factor)

$$\underbrace{\boldsymbol{\mu}_1\mathbf{P}_{2,2}}_{\mathbf{P}_{1,1}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_1\mathbf{P}_{2,1}}_{\dot{\mathbf{P}}_{1,1}}\mathbf{y} = \boldsymbol{\mu}_1\underbrace{\left(\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \mathbf{C}_2\right)}_{\mathbf{h}} \tag{194}$$

such that

$$\dot{\mathbf{P}}_{1,1} = \boldsymbol{\mu}_1\mathbf{P}_{2,1} = \left(\dot{\boldsymbol{\mu}_1\mathbf{P}_{2,2}}\right) = \dot{\boldsymbol{\mu}}_1\mathbf{P}_{2,2} + \boldsymbol{\mu}_1\dot{\mathbf{P}}_{2,2} \tag{195}$$

and by multiplying by $\boldsymbol{\mu}_1^{-1}$ to the left

$$\boldsymbol{\mu}_1^{-1}\dot{\boldsymbol{\mu}}_1 = \left(\mathbf{P}_{2,1} - \dot{\mathbf{P}}_{2,2}\right)\mathbf{P}_{2,2}^{-1} \tag{196}$$

with solution

$$\boldsymbol{\mu}_1 = \exp\left(\int \left(\mathbf{P}_{2,1} - \dot{\mathbf{P}}_{2,2}\right)\mathbf{P}_{2,2}^{-1}\,\mathrm{d}t\right). \tag{197}$$

Again, $\mathbf{P}_{1,1}$ is known after evaluating $\boldsymbol{\mu}_1$ and Eq. (194) can be written as

$$(\dot{\mathbf{P}_{1,1}\mathbf{y}}) = \boldsymbol{\mu}_1\mathbf{h} \tag{198}$$

with solution

$$\mathbf{y} = \mathbf{P}_{1,1}^{-1}\left(\int \boldsymbol{\mu}_1\left\{\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \mathbf{C}_2\right\}\mathrm{d}t + \mathbf{C}_1\right), \tag{199}$$

where $\mathbf{C}_2$ is a constant vector. Complete solution can be split into its particular and homogeneous counterparts as

$$\mathbf{y}_{\mathbf{p}} = \mathbf{P}_{1,1}^{-1}\int \boldsymbol{\mu}_1\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t, \tag{200}$$

and

$$\mathbf{y}_{\mathbf{h}} = \mathbf{P}_{1,1}^{-1}\left(\int \boldsymbol{\mu}_1\mathbf{C}_2\,\mathrm{d}t + \mathbf{C}_1\right). \tag{201}$$

This procedure can be time consuming depending on how matrices **M**, **C** and **K** vary with time. Other issue is the solution of Eq. (191).

It is worth to mention that the proposed procedure does not depend on a time discretization and, despite being solved by using a computer, is analytical. For example, one can evaluate the response $\mathbf{y}(t)$ at any given time without knowing the solution of previous times. Numerical methods, like the well established Newmark-beta method, work in a total different way, building the solution from time step to time step. Approximation errors associated to the interpolation hypothesis of each particular numerical method are sensitive to "large" time steps, such that approximation errors (as well as numerical errors) are expected when using methods relying on approximations.

In the following, the special case of constant coefficients is addressed.

3.2 CONSTANT COEFFICIENTS

Consider a discrete system of n coupled ODEs with constant coefficients. Recalling Eq. (180)

$$\underbrace{\boldsymbol{\mu}_2\mathbf{I}}_{\mathbf{P}_{2,2}}\ddot{\mathbf{y}}+\underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,1}}_{\dot{\mathbf{P}}_{2,2}}\dot{\mathbf{y}}+\underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,2}}_{\mathbf{P}_{2,1}}\dot{\mathbf{y}}+\underbrace{\boldsymbol{\mu}_2\bar{\mathbf{K}}}_{\dot{\mathbf{P}}_{2,1}}\mathbf{y}=\boldsymbol{\mu}_2\bar{\mathbf{f}}. \tag{202}$$

For the following steps, the time-derivatives of $\mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,2}$ are considered null, since these partitions are constant just like matrix $\bar{\mathbf{C}}$. Thus,

$$\dot{\mathbf{P}}_{2,2}=\dot{\boldsymbol{\mu}}_2=\boldsymbol{\mu}_2\mathbf{F}_{2,1,1}. \tag{203}$$

Multiplying to the left by the inverse of the integrating factor yields

$$\boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2=\mathbf{F}_{2,1,1}. \tag{204}$$

The same procedure can be performed to $\mathbf{P}_{2,1}$,

$$\dot{\mathbf{P}}_{2,1}=\boldsymbol{\mu}_2\bar{\mathbf{K}}=\dot{\boldsymbol{\mu}}_2\mathbf{F}_{2,1,2}, \tag{205}$$

which, with further simplifications, yields

$$\boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2=\bar{\mathbf{K}}\mathbf{F}_{2,1,2}^{-1}. \tag{206}$$

Equating Eq. (204) to Eq. (208), and multiplying to the right by $\mathbf{F}_{2,1,2}=\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}$ results in

$$\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]=\bar{\mathbf{K}}, \tag{207}$$

further simplifying this equation yields

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{0}, \tag{208}$$

where $\mathbf{0}$ is the null $n \times n$ matrix. The analytical solution procedure for this matrix polynomial equation is discussed in Subsec. 3.2.1.

The first generalized integrating factor, $\boldsymbol{\mu}_2$, is found by means of Eq. (204)

$$\boldsymbol{\mu}_2 = \exp\left(\int \mathbf{F}_{2,1,1}\,\mathrm{d}t\right) = \exp\left(\mathbf{F}_{2,1,1}t\right). \tag{209}$$

Equation (202) can be written as

$$(\dot{\overline{\mathbf{P}_{2,2}\dot{\mathbf{y}}}}) + (\dot{\overline{\mathbf{P}_{2,1}\mathbf{y}}}) = \boldsymbol{\mu}_2\bar{\mathbf{f}} \tag{210}$$

which can be integrated w.r.t time, yielding

$$\boldsymbol{\mu}_2\dot{\mathbf{y}} + \boldsymbol{\mu}_2\mathbf{F}_{2,1,2}\mathbf{y} = \int \boldsymbol{\mu}_2\bar{\mathbf{f}}\mathrm{d}t + \mathbf{C}_2, \tag{211}$$

where $\mathbf{C}_2$ is a constant vector. This is a system of first order ordinary differential equations. Thus, by multiplying the equation to the left by $\boldsymbol{\mu}_2^{-1}$ and by another generalized integrating factor $\boldsymbol{\mu}_1$, and using $\mathbf{F}_{2,1,2} = \bar{\mathbf{C}} - \mathbf{F}_{2,1,1}$, yields

$$\underbrace{\boldsymbol{\mu}_1\mathbf{I}}_{\mathbf{P}_{1,1}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_1\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]}_{\dot{\mathbf{P}}_{1,1}}\mathbf{y} = \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\mathrm{d}t + \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\mathbf{C}_2. \tag{212}$$

Repeating the same procedure for the second integrating factor,

$$\dot{\mathbf{P}}_{1,1} = \dot{\boldsymbol{\mu}}_1 = \boldsymbol{\mu}_1\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right], \tag{213}$$

or

$$\boldsymbol{\mu}_1^{-1}\dot{\boldsymbol{\mu}}_1 = \bar{\mathbf{C}} - \mathbf{F}_{2,1,1}, \tag{214}$$

such that

$$\boldsymbol{\mu}_1 = \exp\left(\int \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\mathrm{d}t\right) = \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right). \tag{215}$$

Equation (212) can be written as

$$(\dot{\overline{\mathbf{P}_{1,1}\mathbf{y}}}) = \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\mathrm{d}t + \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\mathbf{C}_2, \tag{216}$$

such that by integrating and multiplying to left by $\boldsymbol{\mu}_1^{-1}$ yields

$$\mathbf{y} = \boldsymbol{\mu}_1^{-1} \int \boldsymbol{\mu}_1 \boldsymbol{\mu}_2^{-1} \int \boldsymbol{\mu}_2 \bar{\mathbf{f}} \, \mathrm{d}t \, \mathrm{d}t + \boldsymbol{\mu}_1^{-1} \int \boldsymbol{\mu}_1 \boldsymbol{\mu}_2^{-1} \mathbf{C}_2 \, \mathrm{d}t + \boldsymbol{\mu}_1^{-1} \mathbf{C}_1, \tag{217}$$

where $\mathbf{C}_1$ is another constant vector. As the integrating factors are already known for this case, they can be substituted in previous equation, resulting in

$$\begin{aligned} \mathbf{y} = \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \exp\left(-\mathbf{F}_{2,1,1} t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \, \mathrm{d}t \, \mathrm{d}t \\ + \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \exp\left(-\mathbf{F}_{2,1,1} t\right) \mathbf{C}_2 \, \mathrm{d}t \\ + \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \mathbf{C}_1. \end{aligned} \tag{218}$$

According to (GALLIER, 2011), $\exp(\mathbf{A}t)$ is an exponential map and, for such map, the property $\exp(\mathbf{A})\exp(\mathbf{B}) = \exp(\mathbf{A}+\mathbf{B})$ holds true if $\mathbf{AB} = \mathbf{BA}$ ($\mathbf{A}$ and $\mathbf{B}$ commute). Consequently, if $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute the exponential multiplications in Eq. (513) can be grouped together as

$$\begin{aligned} \mathbf{y} = \exp\left(\left[\mathbf{F}_{\mathbf{2,1,1}} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \, \mathrm{d}t \, \mathrm{d}t + \\ \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \mathbf{C}_2 \, \mathrm{d}t + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \mathbf{C}_1. \end{aligned} \tag{219}$$

If $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is non-singular, the time-derivative of the exponential map can be used to evaluate the integral with the $\mathbf{C}_1$ term in Eq. (219),

$$\begin{aligned} \mathbf{y} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \, \mathrm{d}t \, \mathrm{d}t + \\ \exp\left(-\mathbf{F}_{2,1,1} t\right) \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \mathbf{C}_1, \end{aligned} \tag{220}$$

where, again, there is a particular solution,

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \, \mathrm{d}t \, \mathrm{d}t, \tag{221}$$

and a homogeneous solution,

$$\mathbf{y_h} = \exp\left(-\mathbf{F}_{2,1,1} t\right) \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \mathbf{C}_1, \tag{222}$$

which can be further simplified if $\mathbf{C}_2$ absorbs the matrices multiplication, *i.e.*,

$$\mathbf{y_h} = \exp\left(-\mathbf{F}_{2,1,1} t\right) \mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \mathbf{C}_1, \tag{223}$$

both valid if

$$\bar{\mathbf{C}}\mathbf{F}_{2,1,1} = \mathbf{F}_{2,1,1}\bar{\mathbf{C}} \tag{224}$$

commute. Complete expressions to evaluate constants $\mathbf{C}_1$ and $\mathbf{C}_2$ are developed in Appendix B.1 and an efficient approach to evaluate Eq. (223) is discussed in Appendix B.5.

Commutativity of matrices $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}}$ is studied in the following.

3.2.1 Solving the quadratic equation, Eq. (208)

Assume that the damping matrix, $\mathbf{C}$, is given by Rayleigh damping model, *i.e.*, proportional damping. Thus

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}, \tag{225}$$

normalizing by the mass matrix, one gets

$$\bar{\mathbf{C}} = \alpha\mathbf{M}^{-1}\mathbf{M} + \beta\mathbf{M}^{-1}\mathbf{K} = \alpha\mathbf{I} + \beta\bar{\mathbf{K}}, \tag{226}$$

therefore,

$$\bar{\mathbf{C}}\bar{\mathbf{K}} = \alpha\mathbf{I}\bar{\mathbf{K}} + \beta\bar{\mathbf{K}}^2 = \alpha\bar{\mathbf{K}}\mathbf{I} + \beta\bar{\mathbf{K}}^2 = \bar{\mathbf{K}}\bar{\mathbf{C}}, \tag{227}$$

such that $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute.

As stated before, $\mathbf{F}_{2,1,1}$ is given by the following matrix quadratic equation

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{0}. \tag{228}$$

According to (HIGHAM, 2008), a quadratic equation of the form

$$\mathbf{X}^2 + \mathbf{B}\mathbf{X} + \mathbf{D} = \mathbf{0}, \tag{229}$$

has a solution given by

$$\mathbf{X} = -\frac{1}{2}\mathbf{B} + \frac{1}{2}\left[\mathbf{B}^2 - 4\mathbf{D}\right]^{\frac{1}{2}}, \tag{230}$$

if $\mathbf{B}$ and $\mathbf{D}$ commute. When a matrix solves a matrix polynomial equation it is called a *solvent*. In Eq. (228), the unknown matrix variable is at the left of the coefficient, $\mathbf{F}_{2,1,1}\bar{\mathbf{C}}$, hence, it is called a left solvent (HIGHAM; KIM, 2001), while in Eq. (230) the unknown variable is at the right, **BX**, thus, a right solvent. The question is whether solvent given by Eq. (230) solves Eq. (228), since Eq. (228) has the first coefficient equal to **I** and $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute. To validate it, let the candidate of solvent, $\tilde{\mathbf{F}}_{2,1,1}$, be calculated by Eq. (230) and be substituted into Eq. (228).

$$\tilde{\mathbf{F}}_{2,1,1} = \frac{1}{2}\bar{\mathbf{C}} + \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}, \tag{231}$$

applyied to Eq. 228,

$$\frac{1}{4}\bar{\mathbf{C}}^2 + \frac{1}{4}\bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} + \frac{1}{4}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} + \frac{1}{4}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right] - \tag{232}$$

$$\frac{1}{2}\bar{\mathbf{C}}^2 - \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} + \bar{\mathbf{K}}. \tag{233}$$

The terms without the square roots are cancelled such that

$$\frac{1}{4}\bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} + \frac{1}{4}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} - \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}}. \tag{234}$$

From Corollary 1.34 of (HIGHAM, 2008), a special case of function defined on the spectra of the multiplication of two matrices is the square root function, that gives

$$[\mathbf{AB}]^{\frac{1}{2}}\mathbf{A} = \mathbf{A}[\mathbf{BA}]^{\frac{1}{2}}. \tag{235}$$

Let $\mathbf{A} = \bar{\mathbf{C}}$ and $\mathbf{B} = \bar{\mathbf{C}} - 4\bar{\mathbf{C}}^{-1}\bar{\mathbf{K}}$, thus,

$$\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} = \bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{C}}^{-1}\bar{\mathbf{K}}\bar{\mathbf{C}}\right]^{\frac{1}{2}}; \tag{236}$$

as $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute,

$$\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} = \bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{C}}^{-1}\bar{\mathbf{C}}\bar{\mathbf{K}}\right]^{\frac{1}{2}} = \bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}, \tag{237}$$

hence,

$$\frac{1}{4}\bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} + \frac{1}{4}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} - \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} = \mathbf{0}, \tag{238}$$

therefore Eq. (231) is a solvent to Eq. (228) and

$$\mathbf{F}_{2,1,1} = \frac{1}{2}\bar{\mathbf{C}} + \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}. \tag{239}$$

Another interesting fact is that, as Eq. (231) was derived for a right solvent and satisfies a left solvent as well, such that

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{F}_{2,1,1}^2 - \bar{\mathbf{C}}\mathbf{F}_{2,1,1} + \bar{\mathbf{K}} = \mathbf{0}, \tag{240}$$

thus, by inspection, one can realize that $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ also commute and Eq. (220) is consistent when Rayleigh damping is used.

Other useful relations valid for Rayleigh damping are

$$\bar{\mathbf{C}}\mathbf{F}_{2,1,1} = \left(\alpha\mathbf{I} + \beta\bar{\mathbf{K}}\right)\mathbf{F}_{2,1,1} = \alpha\mathbf{F}_{2,1,1} + \beta\bar{\mathbf{K}}\mathbf{F}_{2,1,1} \tag{241}$$

and

$$\mathbf{F}_{2,1,1}\bar{\mathbf{C}} = \mathbf{F}_{2,1,1}\left(\alpha\mathbf{I} + \beta\bar{\mathbf{K}}\right) = \alpha\mathbf{F}_{2,1,1} + \beta\mathbf{F}_{2,1,1}\bar{\mathbf{K}}, \tag{242}$$

thus, as $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute, by comparing the last two equations if follows that $\bar{\mathbf{K}}$ and $\mathbf{F}_{2,1,1}$ also commute, *i.e.*, $\mathbf{F}_{2,1,1}\bar{\mathbf{K}} = \bar{\mathbf{K}}\mathbf{F}_{2,1,1}$.

Other form of damping leading to commutativity between $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,2}$ is

$$\bar{\mathbf{C}} = \alpha\mathbf{I} + \sum_j \beta_j \bar{\mathbf{K}}^j, \tag{243}$$

for example, where the Rayleigh damping turns out to be a particular case. Equation (243) is known as the Caughey series and is also known to be the condition for a system to have normal modes (ADHIKARI, 2006).

A numerical method shall be used to find $\mathbf{F}_{2,1,1}$ when other models of constant damping fail to keep commutativity between the damping and the stiffness matrices (HIGHAM, 2008).

3.2.2 Under damped problems

Equations (442) and (223) depend on

$$\exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\right) \tag{244}$$

and

$$\exp\left(-\mathbf{F}_{2,1,1}\right). \tag{245}$$

For under-damped problems, it is possible to show that

$$\exp\left(\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right)^* = \exp\left(-\mathbf{F}_{2,1,1}\right). \tag{246}$$

First, as $\exp\left(\mathbf{A}^*\right) = \exp\left(\mathbf{A}\right)^*$ it is possible to write

$$\mathbf{F}_{2,1,1}^* - \bar{\mathbf{C}}^* = -\mathbf{F}_{2,1,1} \tag{247}$$

and as $\bar{\mathbf{C}}$ is strictly real

$$\mathbf{F}_{2,1,1}^{*} = -\mathbf{F}_{2,1,1} + \bar{\mathbf{C}} \tag{248}$$

such that

$$\Re(\mathbf{F}_{2,1,1}) = -\Re(\mathbf{F}_{2,1,1}) + \bar{\mathbf{C}} \implies \Re(\mathbf{F}_{2,1,1}) = \frac{1}{2}\bar{\mathbf{C}} \tag{249}$$

and

$$-\Im(\mathbf{F}_{2,1,1}) = -\Im(\mathbf{F}_{2,1,1}). \tag{250}$$

Thus, for under-damped problems the imaginary part of $\mathbf{F}_{2,1,1}$ should be only related to the second term in the RHS of Eq (239)

$$\Im(\mathbf{F}_{2,1,1}) = \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} \tag{251}$$

and this is true iff the matrix inside the square root is strictly real and has only real negative eigenvalues. The first condition is always fulfilled since both $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ are real matrices. Additionally, the term $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ in Eq. (223) reduces to

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} = -2\Im(\mathbf{F}_{2,1,1})\, i \tag{252}$$

a purely imaginary matrix.

In the following, conditions necessary to satisfy are investigated that

$$\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}} \tag{253}$$

has only real negative eigenvalues.

Structural damping

The simpler form of proportional damping is given by $\mathbf{C} = \beta\mathbf{K}$. In this case, the matrix inside the square root is

$$\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}} = \beta^2\bar{\mathbf{K}}^2 - 4\bar{\mathbf{K}}. \tag{254}$$

It is worth noticing that the traditional eigenvalue problem is

$$(\mathbf{K} - \lambda\mathbf{M})\,\mathbf{x} = \mathbf{0} \tag{255}$$

where λ is a strictly real positive eigenvalue and $\mathbf{x}$ is the associated eigenvector. Pre-multiplying by $\mathbf{M}^{-1}$

$$\left(\bar{\mathbf{K}} - \lambda\mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{256}$$

and multiplying by $\bar{\mathbf{K}}$

$$\bar{\mathbf{K}}^2\mathbf{x} = \lambda\bar{\mathbf{K}}\mathbf{x} = \lambda\lambda\mathbf{x} = \lambda^2\mathbf{x}. \tag{257}$$

Thus, Eq. (254) can also be written as

$$\beta^2\mathbf{X}\mathbf{\Lambda}^2\mathbf{X}^{-1} - 4\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1} \tag{258}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\bar{\mathbf{K}}$ and $\mathbf{X}$ a matrix with its corresponding eigenvectors. Arranging

$$\mathbf{X}\left(\beta^2\mathbf{\Lambda}^2 - 4\mathbf{\Lambda}\right)\mathbf{X}^{-1} \tag{259}$$

such that

$$\beta^2\lambda_i^2 - 4\lambda_i < 0\ \forall i = 1..n \tag{260}$$

and β must satisfy

$$\beta < \sqrt{\frac{4}{\lambda_i}}\ \forall i = 1..n. \tag{261}$$

Proportional Damping

Considering

$$\bar{\mathbf{C}} = \alpha\mathbf{I} + \beta\bar{\mathbf{K}}. \tag{262}$$

The square of $\bar{\mathbf{C}}$ is

$$\bar{\mathbf{C}}^2 = \alpha^2\mathbf{I} + 2\beta\alpha\bar{\mathbf{K}} + \beta^2\bar{\mathbf{K}}^2, \tag{263}$$

such that the term inside the square root is

$$\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}} = \alpha^2\mathbf{I} + 2\beta\alpha\bar{\mathbf{K}} + \beta^2\bar{\mathbf{K}}^2 - 4\bar{\mathbf{K}}. \tag{264}$$

Using the change of basis

$$\mathbf{X}\left(\alpha^2\mathbf{I} + 2\beta\alpha\mathbf{\Lambda} + \beta^2\mathbf{\Lambda}^2 - 4\mathbf{\Lambda}\right)\mathbf{X}^{-1} \tag{265}$$

such that

$$\alpha^2 + 2\beta\alpha\lambda_i + \beta^2\lambda_i^2 - 4\lambda_i < 0\ \forall i = 1..n. \tag{266}$$

There are many possibilities to find suitable values of α and β in the previous equation. On can notice that the last two terms should be negative or zero if β satisfies Eq. (261). If

$$\beta^2\lambda_i^2 - 4\lambda_i = -\varepsilon_i \tag{267}$$

then

$$\bar{\beta} = \sqrt{\frac{-\varepsilon_i + 4\lambda_i}{\lambda_i^2}}. \tag{268}$$

Thus,

$$\alpha^2 + \left(2\bar{\beta}\lambda_i\right)\alpha - \varepsilon_i < 0 \; \forall i = 1..n, \tag{269}$$

such that

$$\bar{\alpha} < -\left(\bar{\beta}\lambda_i\right) \pm \sqrt{\left(\bar{\beta}\lambda_i\right)^2 + \varepsilon_i}. \tag{270}$$

3.3 COMPARISON WITH OTHER SOLUTION PROCEDURES

The proposed approach can be compared with other solution procedures to solve coupled systems of second order ODEs. Among the many options found in the literature, the most used approaches to solve coupled systems of linear ODEs are numerical, like the Newmark-beta method (NOH; BATHE, 2019) and analytical approaches, like the Laplace Transform (WORDU; OJONG; OKPARANMA, 2022) and state variables as a order reduction approach (CHAHANDE; ARORA, 1994). One can also use a mix of reduction order approaches, like the State Variable, and numerical solutions for first order ODEs. The numerical approaches are the most used, since they are generic (do not assume a particular form for the excitation) and are easy to implement. Nonetheless, every numerical method is based on some assumptions on the behavior of the response between discrete time points and, therefore, is prone to different types of errors.

Well established analytical methods, like the Laplace Transform, are hard to use for large dimensions, since the symbolic inverse operations are very complex to perform, even when using modern Computer Algebra Softwares (CAS) like (MAPLESOFT, ; KARJANTO; HUSAIN, 2021). Other well known analytical procedures, like the Variation of Parameters Method depends on the previous knowledge of the homogeneous solution. Also, the authors could only find references addressing its use for first order coupled systems of ODEs (ABELL; BRASELTON, 2023; NAGLE; SAFF; SNIDER, 2000) but not for second-order problems.

3.3.1 Solution using the matrix Laplace transform

According to (NAGLE; SAFF; SNIDER, 2000), the Laplace transform $\mathscr{L}$ can be naturally extended for systems of coupled ODEs by applying the original definition of the transform element-wise. Thus, Equation (177) with constant coefficients becomes

$$\begin{aligned}\mathscr{L}\left(\mathbf{I}\ddot{\mathbf{y}} + \bar{\mathbf{C}}\dot{\mathbf{y}} + \bar{\mathbf{K}}\mathbf{y}\right) = s^2\mathscr{L}(\mathbf{y}) - s\mathbf{u}_0 - \mathbf{v}_0 + \bar{\mathbf{C}}\left[s\mathscr{L}(\mathbf{y}) - \mathbf{u}_0\right] + \bar{\mathbf{K}}\mathscr{L}(\mathbf{y}) = \\ \left[s^2\mathbf{I} + s\bar{\mathbf{C}} + \bar{\mathbf{K}}\right]\mathscr{L}(\mathbf{y}) - \left[s\mathbf{I} + \bar{\mathbf{C}}\right]\mathbf{u}_0 - \mathbf{v}_0 = \mathscr{L}\left(\bar{\mathbf{f}}\right),\end{aligned} \tag{271}$$

where s is the independent variable in the transformation domain, and $\mathbf{u}_0$ and $\mathbf{v}_0$ are initial conditions. Hence, Equation (271) can be further simplified to

$$\mathscr{L}(\mathbf{y}) = \left[s^2\mathbf{I} + s\bar{\mathbf{C}} + \bar{\mathbf{K}}\right]^{-1}\left(\mathscr{L}\left(\bar{\mathbf{f}}\right) + \left[s\mathbf{I} + \bar{\mathbf{C}}\right]\mathbf{u}_0 + \mathbf{v}_0\right). \tag{272}$$

Using matrix inverse properties, the analytical solution is written as

$$\mathbf{y}(t) = \mathscr{L}^{-1}\left(\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]^{-1}(\mathscr{L}(\mathbf{f}) + [s\mathbf{M} + \mathbf{C}]\mathbf{u}_0 + \mathbf{M}\mathbf{v}_0)\right). \tag{273}$$

From Eq. (273), one can observe that the use of Laplace transform to obtain analytical solutions to systems of ODEs is rather complicated, since a symbolic inverse of a matrix of the same dimensionality of the original problem must be calculated and, beyond that, the multiplication of this inverse to the Laplace transform of the excitation vector must be also carried out. Then, algebraic operations are needed to make Eq. (273) suitable for the inverse operation of the Laplace transform. All that said, it is clear that this integral transform method is not suitable for problems with high dimensionality and it is even more cumbersome to find closed-form particular solutions like the ones discussed in the rest of this text.

3.3.2 Solution using order reduction by state variables

Consider the linear coupled systems of second order ODEs given by Eq. (177). Defining a new variable

$$\mathbf{q}(t) = \begin{Bmatrix} \mathbf{y}(t) \\ \dot{\mathbf{y}}(t) \end{Bmatrix}, \tag{274}$$

it is possible to re-write the original system as (PALM, 2020),

$$\underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{A}}\dot{\mathbf{q}}(t) + \underbrace{\begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{K} & \mathbf{C} \end{bmatrix}}_{\mathbf{D}}\mathbf{q}(t) = \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}}_{\mathbf{B}}\mathbf{f}(\mathbf{t}), \tag{275}$$

a first order ODE with twice the original dimension. By normalizing Eq. (275) by matrix $\mathbf{A}$, it becomes

$$\mathbf{I}\dot{\mathbf{q}}(t) + \bar{\mathbf{D}}\mathbf{q}(t) = \bar{\mathbf{B}}\mathbf{f}(t), \tag{276}$$

where $\bar{\mathbf{D}} = \mathbf{A}^{-1}\mathbf{D}$ and $\bar{\mathbf{B}} = \mathbf{A}^{-1}\mathbf{B}$. Analytical solution to Eq. (276) is given by (ABELL; BRASELTON, 2023)

$$\mathbf{q}(t) = \exp\left(-\bar{\mathbf{D}}t\right) \int \exp\left(\bar{\mathbf{D}}t\right) \bar{\mathbf{B}}\mathbf{f}(t)\,\mathrm{d}t + \exp\left(-\bar{\mathbf{D}}t\right) \mathbf{C}_1, \tag{277}$$

in which $\mathbf{C}_1$ is a constant vector.

The main issue with state variables is the doubled dimensionality, which is a particularly big downside of the technique when large problems are to be solved, since the time to evaluate matrix exponentials and matrix products increases non-linearly with the dimensionality. This technique is used as the reference analytical solution to validate the proposed approach and is simply referred as State Variables (SV) in the rest of this text.

3.4 PARTICULAR SOLUTIONS OBTAINED BY CONSIDERING SPECIFIC EXCITATION FUNCTIONS

So far, conditions for obtaining the solution of systems of coupled second order differential equations were discussed. To this end, considering constant coefficients, it was shown that permanent solution is given by Eq. (442). Conditions to solve the quadratic equation associated to $\mathbf{F}_{2,1,1}$ were also discussed in details for proportional damping. Next sections are devoted to discuss further analytical solutions that can be obtained for some particular forms of excitations.

The analytical solutions derived in this work are correct in the sense that they satisfy Eq. (1) for any time t, with no assumption about the behavior of the solution at any other given time. The only assumptions used in the rest of this text are: constant coefficients (linear problems) and proportional damping. The analytical solutions are carefully derived to guarantee the correctness of the solutions and also to help the reader to follow each step.

A common benchmark problem is proposed to evaluate the different formulations obtained in the next sections and to validate the computer implementation. The problem is a 3 DOFs system described by

$$\mathbf{M} = \begin{bmatrix} 2.0 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \tag{278}$$

$$\mathbf{K} = \begin{bmatrix} 6.0 & -4.0 & 0.0 \\ -4.0 & 6.0 & -2.0 \\ 0.0 & -2.0 & 6.0 \end{bmatrix} \times 10^2 \tag{279}$$

and

$$\mathbf{C} = \beta\mathbf{K}, \tag{280}$$

with

$$\mathbf{y}(0) = \dot{\mathbf{y}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{281}$$

where the damping parameter β is specified in each example. This small problem was chosen since it makes easy to visualize the response. Nonetheless, it is a coupled system of ODEs and is able to show all the intended characteristics of the proposed approach.

The analytical solution using the Laplace Transform, Eq. (273), could not be obtained for this problem in closed form using the CAS Maxima (KARJANTO; HUSAIN, 2021) and a simple senoidal excitation. Thus, the order reduction by State variables is used to asses the solutions obtained with the proposed approach. The Newmark-beta method, a reference method in the literature, is also used as reference in one of the examples. The idea is not to use the numerical method to validate the analytical response, but the computer implementation and also to compare the execution times.

3.4.1 Periodic excitations

Let the excitation vector be defined as

$$\mathbf{f}(t) = g_1(t)\mathbf{e}_1 + g_2(t)\mathbf{e}_2 + \ldots + g_n(t)\mathbf{e}_n, \tag{282}$$

where n is the dimension of the problem, *i.e.*, the problem has n degrees of freedom, and $g_j(t)$ is a function of time multiplying unitary vector $\mathbf{e}_j$. Normalizing by the mass matrix

$$\bar{\mathbf{f}} = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1 + \ldots + g_n(t)\mathbf{v}_n. \tag{283}$$

Assuming

$$g_j(t) = \sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right), \tag{284}$$

where n_k is the number of terms, $c_{jk} \in \mathbb{R}$ is an amplitude, $\beta_{jk} = i\omega_{jk} \in \mathbb{C}$ a complex angular frequency and $\phi_{jk} \in \mathbb{C}$ a complex phase.

Substituting this excitation vector into the inner convolution of the particular solution in Eq. (442), one gets

$$\int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \, \mathrm{d}t = \int \exp\left(\mathbf{F}_{2,1,1} t\right) \sum_{j=1}^{n} \sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \mathbf{v}_j \, \mathrm{d}t, \tag{285}$$

as the vectors $\mathbf{v}_j$ are independent of time, they can be left out of the integral to the right side and as the exponentials $c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right)$ are not matrices, they can be commuted with the exponential of matrices,

$$\int \exp\left(\mathbf{F}_{2,1,1}t\right)\, \bar{\mathbf{f}}\,\mathrm{d}t = \sum_{j=1}^{n}\sum_{k=1}^{n_k}\int c_{jk}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\,\mathbf{v}_j. \tag{286}$$

Each one of these integrals can be evaluated by parts

$$\int c_{jk}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right)\mathrm{d}t = \int\left[\frac{c_{jk}}{\beta_{jk}}\left(\exp\left(\beta_{jk}\dot{t}+\phi_{jk}\right)\right)\right]\exp\left(\mathbf{F}_{2,1,1}t\right)\mathrm{d}t =$$
$$\frac{c_{jk}}{\beta_{jk}}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right)\Bigg|_{t_0}^{t} - \int\frac{c_{jk}}{\beta_{jk}}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}\,\mathrm{d}t, \tag{287}$$

where it is possible to neglect the limit value at t_0 as it is implicit in the integration constant $\mathbf{C}_2$.

Grouping common terms and letting $\mathbf{F}_{2,1,1}$ out of the integral at the right side for it being constant

$$\int c_{jk}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\left[\mathbf{I}+\frac{1}{\beta_{jk}}\mathbf{F}_{2,1,1}\right] = \frac{c_{jk}}{\beta_{jk}}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right), \tag{288}$$

such that

$$\int c_{jk}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}\right)t\,\mathrm{d}t = \frac{c_{jk}}{\beta_{jk}}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right)\left[\mathbf{I}+\frac{1}{\beta_{jk}}\mathbf{F}_{2,1,1}\right]^{-1}. \tag{289}$$

Substituting Eq. (289) into Eq. (286),

$$\int \exp\left(\mathbf{F}_{2,1,1}t\right)\, \bar{\mathbf{f}}\,\mathrm{d}t = \sum_{j=1}^{n}\sum_{k=1}^{n_k}\frac{c_{jk}}{\beta_{jk}}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right)\left[\mathbf{I}+\frac{1}{\beta_{jk}}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{v}_j. \tag{290}$$

For the second convolution in Eq. (442),

$$\int \exp\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\mathbf{F}_{2,1,1}t\right)\,\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t = \int\exp\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]t\right)$$
$$\sum_{j=1}^{n}\sum_{k=1}^{n_k}\frac{c_{jk}}{\beta_{jk}}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\mathbf{F}_{2,1,1}t\right)\left[\mathbf{I}+\frac{1}{\beta_{jk}}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{v}_j\,\mathrm{d}t, \tag{291}$$

as $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}}$ commute,

$$\int \exp\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\mathbf{F}_{2,1,1}t\right)\,\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t =$$
$$\int\sum_{j=1}^{n}\sum_{k=1}^{n_k}\frac{c_{jk}}{\beta_{jk}}\exp\left(\beta_{jk}t+\phi_{jk}\right)\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left[\mathbf{I}+\frac{1}{\beta_{jk}}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{v}_j\,\mathrm{d}t. \tag{292}$$

This equation can also be integrated by parts such that

$$\int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \mathrm{d}t \, \mathrm{d}t = \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right)$$
$$\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \left[\mathbf{I} + \frac{1}{\beta_{jk}} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\right]^{-1} \left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j, \tag{293}$$

again, as $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute, the particular solution, Eq. (442), is given by

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_j^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\mathbf{I} + \frac{1}{\omega_j i} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\right]^{-1} \left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j, \tag{294}$$

as $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$,

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right] \left[\mathbf{I} + \frac{1}{\beta_{jk}} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\right]\right]^{-1} \mathbf{v}_j, \tag{295}$$

which simplifies to

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\mathbf{I} + \frac{1}{\beta_{jk}} \bar{\mathbf{C}} - \frac{1}{\beta_{jk}^2} \mathbf{F}_{2,1,1} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\right]^{-1} \mathbf{v}_j, \tag{296}$$

and, using Eq. (207), reduces to

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left(\left[\mathbf{I} + \frac{1}{\beta_{jk}} \bar{\mathbf{C}} - \frac{1}{\beta_{jk}^2} \bar{\mathbf{K}}\right]^{-1} \mathbf{v}_j\right), \tag{297}$$

It is worth noticing that previous expression can be further simplified. First, recall that $\mathbf{v}_j = \mathbf{M}^{-1}\mathbf{e}_j$. Also, as $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$

$$\left[\mathbf{I} + \frac{1}{\beta_{jk}} \bar{\mathbf{C}} + \frac{1}{\beta_{jk}^2} \bar{\mathbf{K}}\right]^{-1} \mathbf{M}^{-1} = \left(\mathbf{M} \left[\mathbf{I} + \frac{1}{\beta_{jk}} \bar{\mathbf{C}} + \frac{1}{\beta_{jk}^2} \bar{\mathbf{K}}\right]\right)^{-1} \tag{298}$$

such that

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\mathbf{M} + \frac{1}{\beta_{jk}} \mathbf{C} + \frac{1}{\beta_{jk}^2} \mathbf{K}\right]^{-1} \mathbf{e}_j. \tag{299}$$

It is also possible to manipulate the term $1/\beta_{j,k}^2$ by inserting it into the matrix

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\beta_{jk}^2 \mathbf{M} + \beta_{jk} \mathbf{C} + \mathbf{K}\right]^{-1} \mathbf{e}_j \tag{300}$$

and, as $\beta_{jk} = i\omega_{jk}$,

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\mathbf{K} + i\omega_{jk}\mathbf{C} - \omega_{jk}^2 \mathbf{M}\right]^{-1} \mathbf{e}_j \tag{301}$$

or

$$\mathbf{y_p}(t) = \sum_{j=1}^{n} \sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \mathbf{K_D}_{jk}^{-1} \mathbf{e}_j, \tag{302}$$

where

$$\mathbf{K_D}_{jk} = \left[\mathbf{K} + i\omega_{jk}\mathbf{C} - \omega_{jk}^2 \mathbf{M}\right] \tag{303}$$

is the Dynamic Stiffness Matrix for jk. It is worth noticing that particular solution given by Eq. (302), a superposition of harmonic responses, can be computed directly using $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ with no need to compute $\mathbf{F}_{2,1,1}$. Solution of Eq. (302) can be written as a linear combination of pre processed vectors $\mathbf{k}_{jk}$

$$\mathbf{K_D}_{jk} \mathbf{k}_{jk} = \mathbf{e}_j, \tag{304}$$

since these operations do not depend on t. Thus, $\mathbf{y_p}(t)$ can be efficiently evaluated as

$$\mathbf{y_p}(t) = \sum_{j=1}^{n} \sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \mathbf{k}_{jk}, \tag{305}$$

for a given time t.

The first derivative of Eq. (302) with respect to time t

$$\dot{\mathbf{y}}_\mathbf{p}(t) = \sum_{j=1}^{n} \sum_{k=1}^{n_k} c_{jk} \beta_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \mathbf{k}_{jk}, \tag{306}$$

is needed to evaluate the constant vectors $\mathbf{C}_1$ and $\mathbf{C}_2$ (Appendix B.1).

When there are complex-conjugate pairs in the β coefficients, an interesting implementation optimization can be made for Eq. (304). Let two matrices $\mathbf{K_D}_{j\xi}$ and $\mathbf{K_D}_{j\nu}$ be the dynamic stiffness matrices for the j-th degree of freedom, and let their β coefficients be complex-conjugate, *i.e.*, $\beta_{j\xi} = \beta_{j\nu}^*$; following from Eq. (303), $\mathbf{K_D}_{j\xi}$ and $\mathbf{K_D}_{j\nu}$ are also complex-conjugate. Thus, $\mathbf{k}_{j\xi}$ can be written as

$$\mathbf{k}_{j\xi} = \mathbf{K}_{\mathbf{D}j\xi}^{-1} \mathbf{e}_j = \left(\mathbf{K}_{\mathbf{D}j\nu}^*\right)^{-1} \mathbf{e}_j, \tag{307}$$

which, using the property of inverse of complex-conjugate matrices, is simplified to

$$\mathbf{k}_{j\xi} = \left(\mathbf{K}_{\mathbf{D}j\nu}^{-1}\right)^* \mathbf{e}_j = \left(\mathbf{K}_{\mathbf{D}j\nu}^{-1} \mathbf{e}_j\right)^* = \mathbf{k}_{j\nu}^*. \tag{308}$$

Thereby, when there are pairs of complex-conjugate β coefficients, just half of the dynamic stiffness matrices must be evaluated.

Example

In mechanical, electrical and civil engineering applications, periodic functions appear, both analytically or represented using Fourier series expansion. These problems arise due to harmonic excitation, examples range from cyclic forces, like support movement and wind drag, up to electromagnetic loads and alternate current. Thus, finding particular solutions to such problems is paramount for simulating and optimizing behavior of these structures. An example of a system with 3 degrees of freedom is taken from (KELLY, 2000) to illustrate the use of such excitations with the proposed approach. The previously presented matrices are used with the excitation given by

$$\mathbf{f} = \begin{Bmatrix} 0 \\ 3sin(4t) \\ 0 \end{Bmatrix}, \tag{309}$$

and $\beta = 1 \times 10^{-2}$. This is an under damped harmonic problem with known permanent solution

$$\mathbf{y_p} = [\mathbf{K} + 4i\mathbf{C} - 16\mathbf{M}]^{-1} \begin{Bmatrix} 0 \\ 3 \\ 0 \end{Bmatrix} \tag{310}$$

with amplitude

$$|\mathbf{y_p}| = \begin{Bmatrix} 9.687 \\ 13.756 \\ 4.711 \end{Bmatrix} \times 10^{-3}, \tag{311}$$

such that

$$\mathbf{y_p}(t) = \sin(4t)|\mathbf{y_p}|. \tag{312}$$

Using the proposed formulation,

$$\mathbf{f} = g_2(t)\mathbf{e}_2 = 3\sin(4t) \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \tag{313}$$

where, by using Euller's identity,

$$g_2(t) = 3\sin(4t) = \frac{3i}{2}\exp(-4it) - \frac{3i}{2}\exp(4it) \tag{314}$$

and $n_k = 2$, $c_{21} = \frac{3i}{2}$, $\omega_{21} = -4$, $\beta_{21} = -4i$, $c_{22} = \frac{-3i}{2}$, $\omega_{22} = 4$, $\beta_{22} = 4i$.

Particular solution can then be found by using Eq. (302)

$$\mathbf{y_p} = \sum_{j=2}^{2}\sum_{k=1}^{2} c_{jk} \exp\left(\beta_{jk} t\right) \mathbf{K_D}_{jk}^{-1} \mathbf{e}_j = \frac{3i}{2} \exp\left(-4it\right) \mathbf{K_D}_{21}^{-1} \mathbf{e}_2 + \frac{-3i}{2} \exp\left(4it\right) \mathbf{K_D}_{22}^{-1} \mathbf{e}_2 \quad (315)$$

with

$$\mathbf{K_D}_{21} = \mathbf{K} - 4i\mathbf{C} - 16\mathbf{M}, \quad (316)$$

and

$$\mathbf{K_D}_{22} = \mathbf{K} + 4i\mathbf{C} - 16\mathbf{M}. \quad (317)$$

Term $\mathbf{z} = \mathbf{K_D}_{21}^{-1}\mathbf{e}_2$ is conjugate to $\mathbf{K_D}_{22}^{-1}\mathbf{e}_2$. Thus,

$$\mathbf{y_p}(t) = 3\left(\frac{i}{2}\exp(-4it)\mathbf{z} - \frac{i}{2}\exp(4it)\mathbf{z}^*\right) \quad (318)$$

or, by splitting $\mathbf{z}$ and $\mathbf{z}^*$ into their real and imaginary parts

$$\mathbf{y_p}(t) = 3\left(\frac{i}{2}\exp(-4it)\Re(\mathbf{z}) + \frac{i^2}{2}\exp(-4it)\Im(\mathbf{z}) - \frac{i}{2}\exp(4it)\Re(\mathbf{z}) + \frac{i^2}{2}\exp(4it)\Im(\mathbf{z})\right). \quad (319)$$

Collecting common terms

$$\mathbf{y_p}(t) = \underbrace{3\left(\frac{i}{2}\exp(-4it) - \frac{i}{2}\exp(4it)\right)}_{\sin(4t)} \Re(\mathbf{z}) + \underbrace{3\left(\frac{i}{2}\exp(-4it) + \frac{i}{2}\exp(4it)\right)}_{0} \Im(\mathbf{z}) \quad (320)$$

such that

$$\mathbf{y_p}(t) = \underbrace{3\frac{i}{2}\left(\exp\left(-4it\right) - \exp\left(4it\right)\right)}_{\sin(4t)} \begin{Bmatrix} 3.229 \\ 4.585 \\ 1.570 \end{Bmatrix} \times 10^{-3} \quad (321)$$

or

$$\mathbf{y_p}(t) = \sin(4t) \begin{Bmatrix} 9.686 \\ 13.756 \\ 4.711 \end{Bmatrix} \times 10^{-3}, \quad (322)$$

the expected solution.

One can verify some conclusions obtained in previous sections. Let start by computing

$$\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}} = \begin{bmatrix} -1187 & 788 & 2 \\ 788 & -1185 & 391 \\ 4 & 782 & -2362 \end{bmatrix} \quad (323)$$

which has real negative eigenvalues -2653.11, -1762.11 and -318.77. Thus, its square root is a complex matrix, as discussed in previous sections

$$\frac{1}{2}\left[\bar{\mathbf{C}}^2-4\bar{\mathbf{K}}\right]^{\frac{1}{2}}=\begin{bmatrix}16.026 & -6.29 & -0.410\\ -6.29 & 15.62 & -2.53\\ -0.820 & -5.06 & 24.03\end{bmatrix}i. \tag{324}$$

Matrix $\bar{\mathbf{K}}$ has eigenvalues

$$\mathbf{\Lambda}=\begin{bmatrix}79.85 & 0.0 & 0.0\\ 0.0 & 445.49 & 0.0\\ 0.0 & 0.0 & 674.66\end{bmatrix} \tag{325}$$

such that inequalities given by Eq. (261) for $\beta = 1\times10^{-2}$ are also satisfied

$$1\times10^{-2}<\left\{\sqrt{\frac{4}{79.85}},\sqrt{\frac{4}{445.49}},\sqrt{\frac{4}{674.66}}\right\}. \tag{326}$$

Matrix

$$\mathbf{F}_{2,1,1}=\frac{1}{2}\bar{\mathbf{C}}+\frac{1}{2}\left[\bar{\mathbf{C}}^2-4\bar{\mathbf{K}}\right]^{\frac{1}{2}}=\begin{bmatrix}1.5 & -1.0 & 0.0\\ -1.0 & 1.5 & -0.5\\ 0.0 & -1.0 & 3.0\end{bmatrix}+\begin{bmatrix}16.026 & -6.29 & -0.410\\ -6.29 & 15.62 & -2.53\\ -0.820 & -5.06 & 24.03\end{bmatrix}i \tag{327}$$

and one can verify that $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}$ are complex-conjugate.

Homogeneous solution, Eq. (223), for this problem is

$$\mathbf{y_h}=\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2+\exp\left(\left(\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right)t\right)\mathbf{C}_1, \tag{328}$$

where

$$\mathbf{C}_1=\begin{Bmatrix}2.49\\ 3.38\\ 1.18\end{Bmatrix}\times10^{-4}+i\begin{Bmatrix}2.36\\ 2.87\\ 1.08\end{Bmatrix}\times10^{-3}, \tag{329}$$

and $\mathbf{C}_2=\mathbf{C}_1^*$ are obtained by solving Eqs. (641) and (642). The maximum order of magnitude of the complex part of the homogeneous solution was 10^{-19}, therefore, the homogeneous solution is real-valued, as expected. Exponentials in Eq. (328) can be computed only once if a constant time step Δt is used, as discussed in Appendix B.5.

The real part of each DOFs of the complete solution $\mathbf{y}(t)=\mathbf{y_h}(t)+\mathbf{y_p}(t)$ is shown in Fig. 9 (solid lines). The solution $\mathbf{y}$ is compared to $\tilde{\mathbf{y}}$ obtained by using traditional Newmark-beta method with standard parameters and $\Delta t = 0.001$s (dotted lines). It is clear that both solutions match.

The solution using State Variables, given by Eq, (277), can be particularized for this examples as

$$\mathbf{y}_{SV}(t) = 3\left[16\mathbf{I}+\bar{\mathbf{D}}^2\right]^{-1}\left[\bar{\mathbf{D}}sin(4t)-4\mathbf{I}cos(4t)\right]\bar{\mathbf{B}}\mathbf{e}_2+\exp\left(-\bar{\mathbf{D}}t\right)\mathbf{C}_{1,SV}, \tag{330}$$

where $\mathbf{C}_{1,SV}$ is a constant vector due to the initial conditions. From Eq. (330) one can observe how such squaring and inverse operations become costlier with the doubled dimensionality associated to the State Variables approach. This solution is compared to the solution given by the generalized integrating factor in Fig. 9.

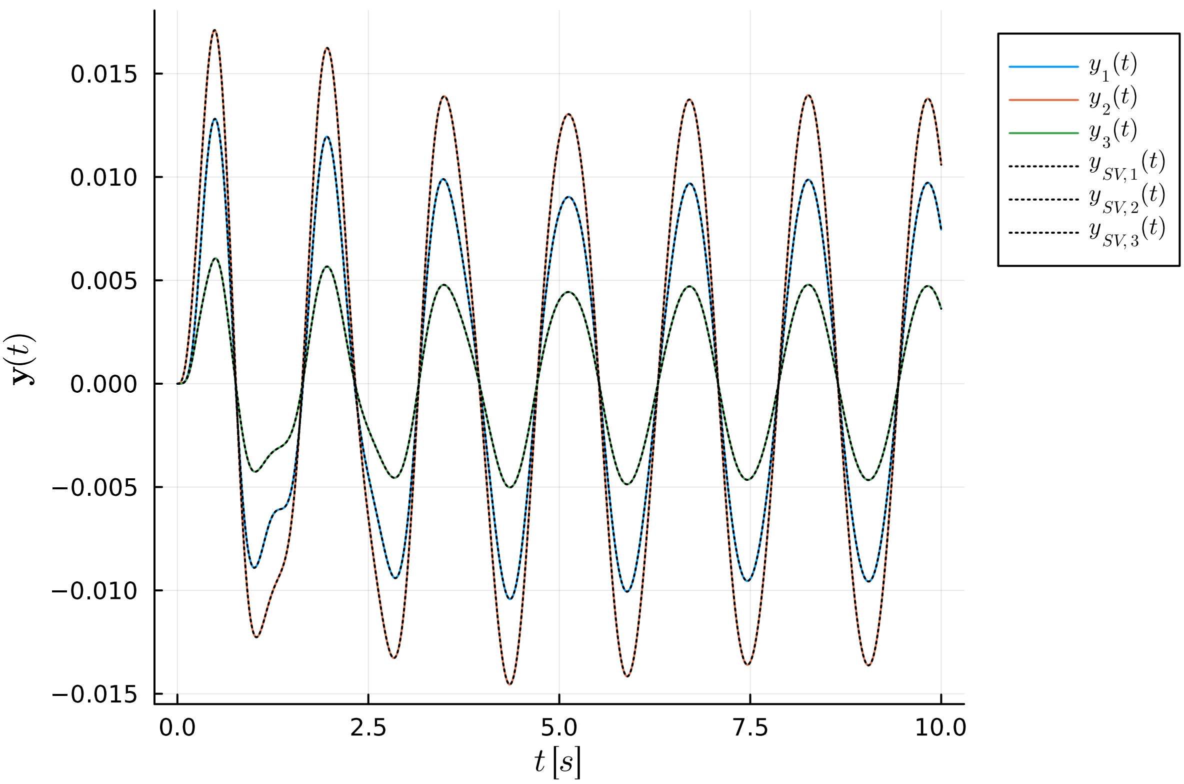

Figure 9 – Complete solution for the under damped problem with sinusoidal excitation. Solutions y_1, y_2 and y_3, (real parts) obtained by using the proposed approach, are shown as solid lines. Solutions $y_{SV,1}$, $y_{SV,2}$ and $y_{SV,3}$, obtained by using State Variables, are shown as dotted lines.

Example - Over Damped System

Let study the previous example with $\beta = 10$, an over damped problem, which arise in many viscous problems in mechanical engineering and highly resistive circuits in electric engineering. The proposed approach does not make any assumption on the level of damping such that the solution procedure used in the previous example does not change. The main differences are the fact that $\mathbf{F}_{2,1,1}$ is a real matrix

$$\mathbf{F}_{2,1,1} = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -2 & 6 \end{bmatrix} \times 10^3, \tag{331}$$

as well as the integration constants

$$\mathbf{C}_1 = \begin{Bmatrix} 1.87 \\ 2.81 \\ 0.94 \end{Bmatrix} \times 10^{-4} \quad (332)$$

and

$$\mathbf{C}_2 = -\begin{Bmatrix} 5.61 \\ 6.23 \\ 2.40 \end{Bmatrix} \times 10^{-9}. \quad (333)$$

The solution using State Variables is the same from the last example, which is again compared to the solution given by the generalized integrating factor in Fig. 10. Hence, the same comments regarding the cost of matrix operations still holds true.

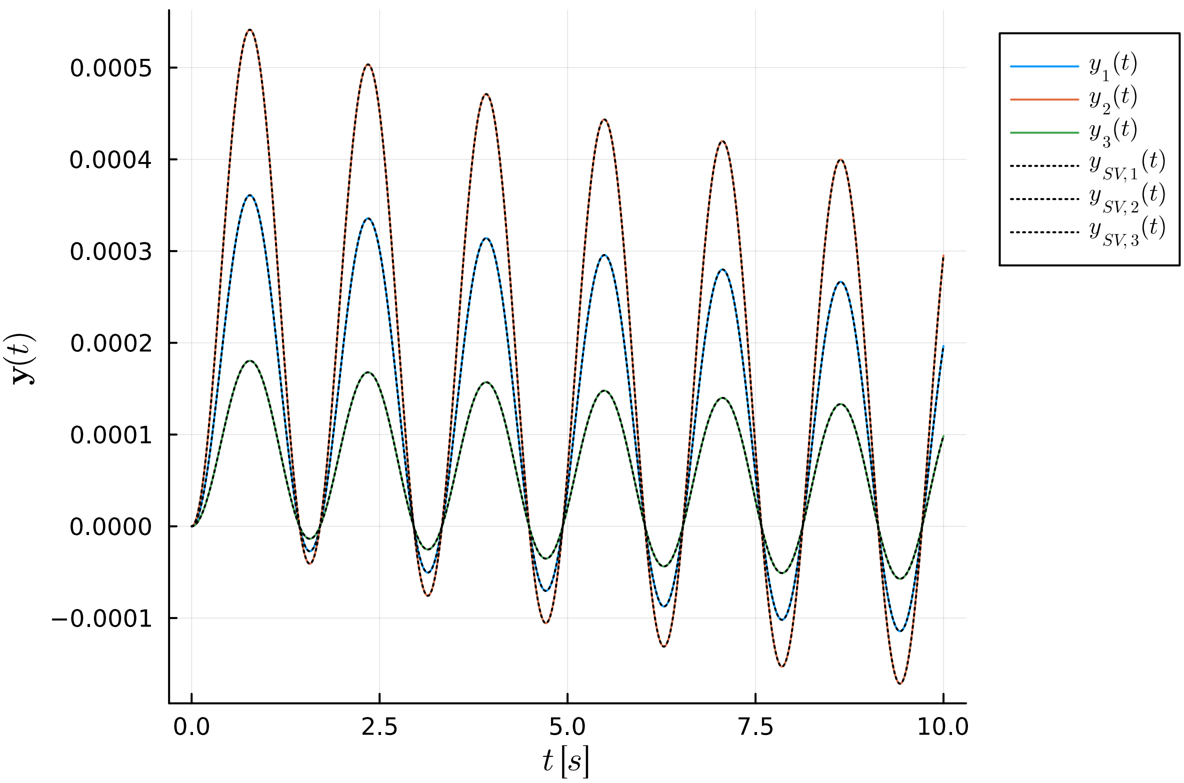

Figure 10 – Complete solution for the over damped problem with sinusoidal excitation. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $y_{SV,1}$, $y_{SV,2}$ and $y_{SV,3}$, obtained by using State Variables, are shown as dotted lines.

3.4.2 Polynomial excitation

Let the normalized excitation vector be defined as

$$\bar{\mathbf{f}} = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1 + \ldots + g_n(t)\mathbf{v}_n, \quad (334)$$

where $g_j(t)$ are polynomial functions

$$g_j(t) = \sum_{k=0}^{n_k} c_{jk} \left(t - t_j\right)^k, \tag{335}$$

$c_{jk} \in \mathbb{R}$ are coefficients, n_k the number of terms and $t_j \in \mathbb{R}$ time shift.

Substituting Eq. (334) into Eq. (442), yields

$$\mathbf{y_p} = e^{\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]t\right) \int \exp\left(\mathbf{F}_{2,1,1}t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \left(t - t_j\right)^k \mathbf{v}_j \, \mathrm{d}t \, \mathrm{d}t, \tag{336}$$

using the linearity of the integral operator and as $\mathbf{v}_j$ does not depend on time results in

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]t\right) \int \exp\left(\mathbf{F}_{2,1,1}t\right) \left(t - t_j\right)^k \mathrm{d}t \, \mathrm{d}t \, \mathbf{v}_j. \tag{337}$$

The inner convolution can be evaluated by parts (here e^x is used instead of $\exp(x)$ to shorten the equations),

$$\begin{aligned}
\int e^{\mathbf{A}t} \left(t - t_j\right)^\alpha \mathrm{d}t &= \int \left(\dot{e^{\mathbf{A}t}\mathbf{A}^{-1}}\right) \left(t - t_j\right)^\alpha \mathrm{d}t \\
&= \left(t - t_j\right)^\alpha e^{\mathbf{A}t}\mathbf{A}^{-1} - \int e^{\mathbf{A}t} \left(t - t_j\right)^{\alpha-1} \mathrm{d}t \, \alpha \mathbf{A}^{-1} \\
&= \left(t - t_j\right)^\alpha e^{\mathbf{A}t}\mathbf{A}^{-1} - \alpha \left(t - t_j\right)^{\alpha-1} e^{\mathbf{A}t}\mathbf{A}^{-2} + \int e^{\mathbf{A}t} \left(t - t_j\right)^{\alpha-2} \mathrm{d}t \, \alpha (\alpha - 1) \mathbf{A}^{-2} \\
&= \left(t - t_j\right)^\alpha e^{\mathbf{A}t}\mathbf{A}^{-1} - \alpha \left(t - t_j\right)^{\alpha-1} e^{\mathbf{A}t}\mathbf{A}^{-2} + \alpha(\alpha-1)\left(t - t_j\right)^{\alpha-2} e^{\mathbf{A}t}\mathbf{A}^{-3} + \cdots + \\
&\quad (-1)^{m-1} \alpha(\alpha-1)\ldots(\alpha-m+2)\left(t - t_j\right)^{\alpha-m+1} e^{\mathbf{A}t}\mathbf{A}^{-m} + \\
&\quad (-1)^m \int e^{\mathbf{A}t} \left(t - t_j\right)^{\alpha-m} \mathrm{d}t \, \alpha(\alpha-1)\ldots(\alpha-m+1)\mathbf{A}^{-m} \\
&\overset{m=\alpha}{=} \left(t - t_j\right)^\alpha e^{\mathbf{A}t}\mathbf{A}^{-1} - \alpha \left(t - t_j\right)^{\alpha-1} e^{\mathbf{A}t}\mathbf{A}^{-2} + \alpha(\alpha-1)\left(t - t_j\right)^{\alpha-2} e^{\mathbf{A}t}\mathbf{A}^{-3} + \cdots + \\
&\quad (-1)^{\alpha-1} \alpha! \left(t - t_j\right) e^{\mathbf{A}t}\mathbf{A}^{-\alpha} + (-1)^\alpha \alpha! e^{\mathbf{A}t}\mathbf{A}^{-\alpha-1},
\end{aligned} \tag{338}$$

which, by arranging all the terms in a single sum, yields

$$\int \exp\left(\mathbf{A}t\right) \left(t - t_j\right)^\alpha \mathrm{d}t = \exp\left(\mathbf{A}t\right) \sum_{l=1}^{\alpha+1} (-1)^{l+1} \frac{\alpha!}{(\alpha - l + 1)!} \left(t - t_j\right)^{\alpha-l+1} \mathbf{A}^{-l}. \tag{339}$$

Using this integral formula with Eq. (337) and the considering that $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute,

$$\begin{aligned}
\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right) \\
\sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k - l + 1)!} \left(t - t_j\right)^{k-l+1} \mathbf{F}_{2,1,1}^{-l} \, \mathrm{d}t \, \mathbf{v}_j.
\end{aligned} \tag{340}$$

As the integral is a linear operator

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \left(t - t_j\right)^{k-l+1} \mathbf{F}_{2,1,1}^{-l} \, \mathrm{d}t \, \mathbf{v}_j, \tag{341}$$

and using again Eq. (339),

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!} \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right)$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-p} \mathbf{F}_{2,1,1}^{-l} \mathbf{v}_j. \tag{342}$$

Since $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute for Rayleigh damping, the matrix exponentials can be cancelled. From Eq. (207), it follows that $\bar{\mathbf{C}} - \mathbf{F}_{2,1,1} = \mathbf{F}_{2,1,1}^{-1} \bar{\mathbf{K}}$. Substituting this relation and applying the inverse property $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$ results in

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{F}_{2,1,1}^{l} \mathbf{F}_{2,1,1}^{-p} \bar{\mathbf{K}}^{p}\right]^{-1} \mathbf{v}_j, \tag{343}$$

and as $\mathbf{v}_j = \mathbf{M}^{-1}\mathbf{e}_j$ and using again the inverse property,

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\mathbf{F}_{2,1,1}^{l-p} \bar{\mathbf{K}}^{p}\right]^{-1} \mathbf{e}_j. \tag{344}$$

As $\bar{\mathbf{K}}$ and $\mathbf{F}_{2,1,1}$ commute for Rayleigh damping, Eq. (241), and p and l are integers, property in Eq. (643) holds,

$$\mathbf{y_p}(t) = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^{p}\mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j. \tag{345}$$

Derivative of previous equation with respect t is needed to evaluate constant vectors $\mathbf{C}_1$ and $\mathbf{C}_2$ (Appendix B.1)

$$\dot{\mathbf{y}}_{\mathbf{p}}(t) = \sum_{j=1}^{n}\sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!}(k-l-p+2)\left(t-t_j\right)^{k-l-p+1}\left[\mathbf{M}\bar{\mathbf{K}}^{p}\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j, \quad (346)$$

valid when $k-l-p+2$ is not zero.

Example

Polynomial excitation is particularly important for electrical engineering simulation, since it can model important functions, like the ramp function for instance. Nevertheless, polynomials can be used to represent variable forces and other phenomena, since even sets of points might be interpolated using polynomials. Thus, consider the 3 DOF problem with $\beta = 1\times 10^{-6}$ and excitation

$$\mathbf{f} = \begin{Bmatrix} 0 \\ 10t - t^2 \\ 0 \end{Bmatrix}, \quad (347)$$

such that $g_2(t) = c_{20} + c_{21}t + c_{22}t^2$ with $c_{20} = 0$, $c_{21} = 10$, $c_{22} = -1$ and $t_2 = 0$.

Evaluation of particular solution $\mathbf{y}_{\mathbf{p}}(t)$, Eq. (345), is shown by means of Alg. 2 where $j = 2$ and $n_k = 2$. The terms obtained in the Algorithm result in

$$\mathbf{y}_{\mathbf{p}}(t) = (2t-10)\left(\left(\mathbf{M}\bar{\mathbf{K}}^2\mathbf{F}_{2,1,1}^{-1}\right)^{-1} + \left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1}\right)\mathbf{e}_2 + \left(10t-t^2\right)\left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1}\mathbf{e}_2$$
$$-2\left(\mathbf{M}\bar{\mathbf{K}}^2\right)^{-1}\mathbf{e}_2 - 2\left(\mathbf{M}\bar{\mathbf{K}}^3\mathbf{F}_{2,1,1}^{-2}\right)^{-1}\mathbf{e}_2 - 2\left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}^{2}\right)^{-1}\mathbf{e}_2. \quad (348)$$

The derivative of the particular response w.r.t time t is given by

$$\dot{\mathbf{y}}_{\mathbf{p}} = 2\left(\left(\mathbf{M}\bar{\mathbf{K}}^2\mathbf{F}_{2,1,1}^{-1}\right)^{-1} + \left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1}\right)\mathbf{e}_2 + (10-2t)\left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1}\mathbf{e}_2. \quad (349)$$

Both expressions can be simplified using matrix inverse properties,

$$\mathbf{y}_{\mathbf{p}}(t) = (2t-10)\left(\mathbf{F}_{2,1,1}\left(\mathbf{M}\bar{\mathbf{K}}^2\right)^{-1} + \left(\mathbf{K}\mathbf{F}_{2,1,1}\right)^{-1}\right)\mathbf{e}_2 + \left(10t-t^2\right)\mathbf{K}^{-1}\mathbf{e}_2 -$$
$$2\left(\mathbf{M}\bar{\mathbf{K}}^2\right)^{-1}\mathbf{e}_2 - 2\mathbf{F}_{2,1,1}^{2}\left(\mathbf{M}\bar{\mathbf{K}}^3\right)^{-1}\mathbf{e}_2 - 2\left(\mathbf{K}\mathbf{F}_{2,1,1}^{2}\right)^{-1}\mathbf{e}_2, \quad (350)$$

and

$$\dot{\mathbf{y}}_{\mathbf{p}} = 2\left(\mathbf{F}_{2,1,1}\left(\mathbf{M}\bar{\mathbf{K}}^2\right)^{-1} + \left(\mathbf{K}\mathbf{F}_{2,1,1}\right)^{-1}\right)\mathbf{e}_2 + (10-2t)\,\mathbf{K}^{-1}\mathbf{e}_2. \quad (351)$$

such that by using Eqs. (641) and (642), one gets the integration constants,

$$\mathbf{C}_1 = \begin{Bmatrix} 8.8792 \\ 0.1457 \\ 4.6479 \end{Bmatrix} \times 10^{-5} + i \begin{Bmatrix} 1.5639 \\ 1.9436 \\ 0.7237 \end{Bmatrix} \times 10^{-3}, \quad (352)$$

with $\mathbf{C}_2 = \mathbf{C}_1^*$.

The solution using State Variables, given by Eq. (277), is particularized for this example as

$$\mathbf{y}_{SV}(t) = \left[-\bar{\mathbf{D}}^{-1}t^2 + 2\bar{\mathbf{D}}^{-2}t + 10\bar{\mathbf{D}}^{-1}t - 2\bar{\mathbf{D}}^{-3} - 10\bar{\mathbf{D}}^{-2}\right]\bar{\mathbf{B}}\mathbf{e}_2 + \exp\left(-\bar{\mathbf{D}}t\right)\mathbf{C}_{1,SV}, \quad (353)$$

where $\mathbf{C}_{1,SV}$ is a vector of integration constants associated to the initial conditions. Although both, Eq. (350) and Eq. (353), present costly matrix operations, one will observe that such operations become much costlier as the size of the matrices increases, as well the fact that the matrices associated to State Variables have twice the dimension. This solution is compared to the solution given by the generalized integrating factor in Fig. 11.

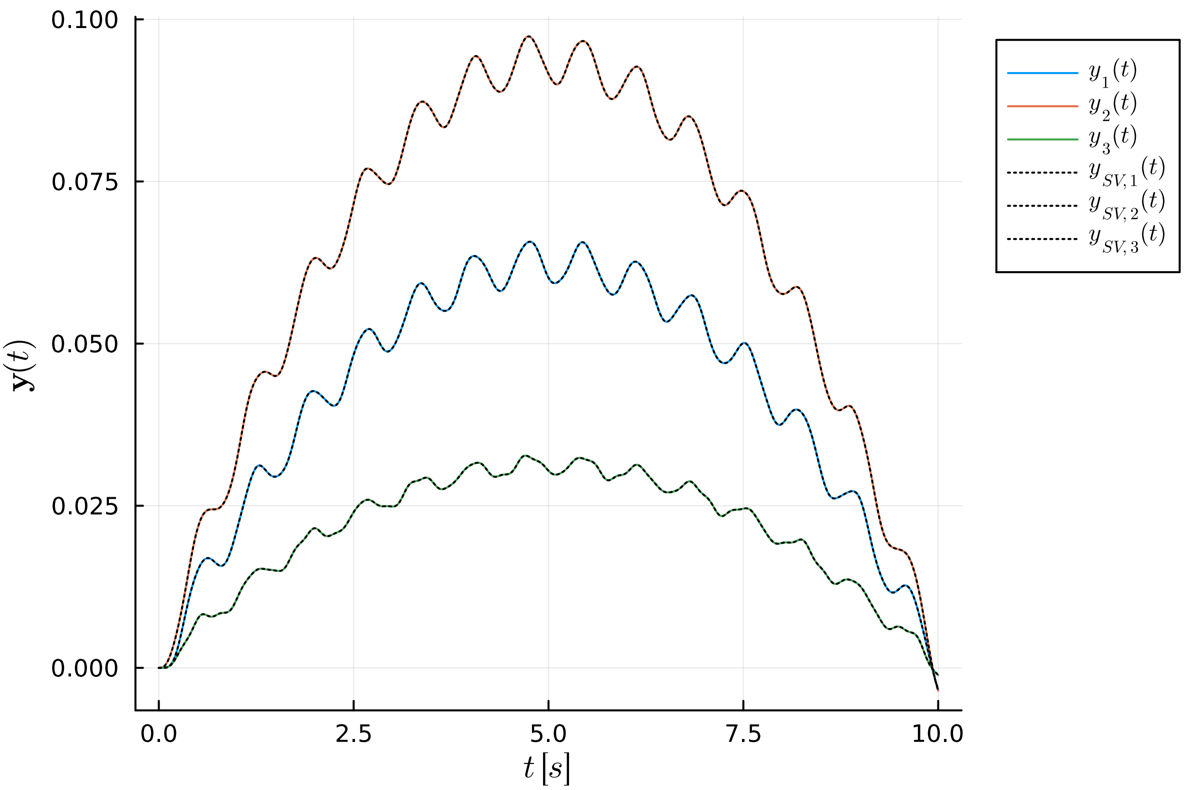

Figure 11 – Complete solution for the three DOFs example subjected to a polynomial excitation. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $y_{SV,1}$, $y_{SV,2}$ and $y_{SV,3}$ obtained by using State Variables are shown as black dotted lines.

Algorithm 2: Evaluation of Eq. (345) for $j=2$ and $n_k=2$.

$k=0$

$c_{1,0}=0$

$k=1$

$c_{1,1}=10$

$l=1 \implies (-1)^{l+1}\frac{k!}{(k-l+1)!}=1$

$p=1 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j = t\left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1}\mathbf{e}_2$

$p=2 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j =$
$-\left(\mathbf{M}\bar{\mathbf{K}}^2\mathbf{F}_{2,1,1}^{-1}\right)^{-1}\mathbf{e}_2$

$l=2 \implies (-1)^{l+1}\frac{k!}{(k-l+1)!}=-1$

$p=1 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j =$
$\left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1}\mathbf{e}_2$

$k=2$

$c_{1,2}=-1$

$l=1 \implies (-1)^{l+1}\frac{k!}{(k-l+1)!}=1$

$p=1 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j = t^2\left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1}\mathbf{e}_2$

$p=2 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j =$
$-2t\left(\mathbf{M}\bar{\mathbf{K}}^2\mathbf{F}_{2,1,1}^{-1}\right)^{-1}\mathbf{e}_2$

$p=3 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j =$
$2\left(\mathbf{M}\bar{\mathbf{K}}^3\mathbf{F}_{2,1,1}^{-2}\right)^{-1}\mathbf{e}_2$

$l=2 \implies (-1)^{l+1}\frac{k!}{(k-l+1)!}=-2$

$p=1 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j =$
$t\left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1}\mathbf{e}_2$

$p=2 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j = -\left(\mathbf{M}\bar{\mathbf{K}}^2\right)^{-1}\mathbf{e}_2$

$l=3 \implies (-1)^{l+1}\frac{k!}{(k-l+1)!}=2$

$p=1 \implies (-1)^{p+1}\frac{(k-l+1)!}{(k-l-p+2)!}\left(t-t_j\right)^{k-l-p+2}\left[\mathbf{M}\bar{\mathbf{K}}^p\mathbf{F}_{2,1,1}^{l-p}\right]^{-1}\mathbf{e}_j =$
$\left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}^2\right)^{-1}\mathbf{e}_2$

3.4.3 Dirac's delta distribution

Let the normalized excitation vector be defined as

$$\bar{\mathbf{f}} = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1+\ldots+g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1+\ldots+g_n(t)\mathbf{v}_n, \tag{354}$$

where $g_j(t)$ are Dirac's delta distributions

$$g_j(t) = \sum_{k=0}^{n_k} c_{jk}\delta\left(t - t_{jk}\right), \tag{355}$$

$c_{jk} \in \mathbb{R}$ are coefficients, n_k the number of terms and $t_{jk} \in \mathbb{R}$ time shifts.

Substituting Eq. (355) into Eq. (442), yields

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk}\delta\left(t - t_{jk}\right) \mathbf{v}_j \,\mathrm{d}t\,\mathrm{d}t, \tag{356}$$

using the linearity of the integral operator and as $\mathbf{v}_j$ does not depend on time results in

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \delta\left(t - t_{jk}\right) \mathrm{d}t\,\mathrm{d}t\,\mathbf{v}_j. \tag{357}$$

Equation (633), Appendix A.3, is used to evaluate the inner convolution

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \exp\left(\mathbf{F}_{2,1,1} t_k\right) \mathscr{H}\left(t - t_{jk}\right) \mathrm{d}t\,\mathbf{v}_j, \tag{358}$$

where $\mathscr{H}$ is the Heaviside step function defined in Eq. (120). As $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,1}$ commute, the two exponential maps can be simplified to

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t + \mathbf{F}_{2,1,1} t_k\right) \mathscr{H}\left(t - t_{jk}\right) \mathrm{d}t\,\mathbf{v}_j, \tag{359}$$

hence, by using the relation derived in Chapter 2

$$\int_0^t \exp\left(\mathbf{A}t\right) f_k(t) \mathscr{H}(t - t_k)\,\mathrm{d}t = \left(\int_{t_k}^t \exp\left(\mathbf{A}t\right) f_k(t)\,\mathrm{d}t\right) \mathscr{H}(t - t_k), \tag{360}$$

where $\mathbf{A}$ is a matrix. The outer convolution is evaluated to

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \mathscr{H}\left(t - t_{jk}\right) \int_{t_k}^{t} \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t + \mathbf{F}_{2,1,1} t_k\right) \mathrm{d}t\,\mathbf{v}_j. \tag{361}$$

If $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is invertible, then

$$\begin{aligned} \mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \mathscr{H}\left(t - t_{jk}\right) \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \left[\exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t + \mathbf{F}_{2,1,1} t_k\right)\right. \\ \left. - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t_k\right)\right] \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j, \end{aligned} \tag{362}$$

otherwise, the integral in Eq. (361) can be evaluated by using the Jordan canonical form (HIGHAM, 2008).

The commutativity of the first power is checked,

$$\begin{gathered}\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]t+\mathbf{F}_{2,1,1}t_k\right)= \\ t^2\left(\mathbf{F}_{2,1,1}\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}^2-\bar{\mathbf{C}}^2+2\bar{\mathbf{C}}\mathbf{F}_{2,1,1}\right)+tt_k\left(\mathbf{F}_{2,1,1}^2-\bar{\mathbf{C}}\mathbf{F}_{2,1,1}\right)\end{gathered} \quad (363)$$

$$\begin{gathered}\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]+\mathbf{F}_{2,1,1}t_k\right)\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t= \\ t^2\left(\bar{\mathbf{C}}\mathbf{F}_{2,1,1}-2\mathbf{F}_{2,1,1}^2-\bar{\mathbf{C}}^2+2\mathbf{F}_{2,1,1}\bar{\mathbf{C}}\right)+tt_k\left(\mathbf{F}_{2,1,1}^2-\mathbf{F}_{2,1,1}\bar{\mathbf{C}}\right),\end{gathered} \quad (364)$$

as $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute, Equation (363) and Equation (364) are equal. Commutativity for the other power is straightforward, since a matrix commutes with itself regardless of different constants multiplying it. Hence, also substituting $\mathbf{v}_j=\mathbf{M}^{-1}\mathbf{e}_j$,

$$\begin{aligned}\mathbf{y_p}=\sum_{j=1}^{n}\Bigg(\sum_{k=0}^{n_k}c_{jk}\mathscr{H}\left(t-t_{jk}\right)[&\exp\left(-\mathbf{F}_{2,1,1}t+\mathbf{F}_{2,1,1}t_k\right) \\ &-\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t_k\right)]\Bigg)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{M}^{-1}\mathbf{e}_j,\end{aligned} \quad (365)$$

rearranging the terms and applying the property $\mathbf{A}^{-1}\mathbf{B}^{-1}=(\mathbf{BA})^{-1}$ results in

$$\begin{aligned}\mathbf{y_p}=\sum_{j=1}^{n}\Bigg(\sum_{k=0}^{n_k}c_{jk}\mathscr{H}\left(t-t_{jk}\right)[&\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right) \\ &-\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t\right)\right)]\Bigg)\left[\mathbf{C}-2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{e}_j.\end{aligned} \quad (366)$$

In Eq. (247), it was shown that, for under-damped problems, $\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}$ is the complex-conjugate of $\mathbf{F}_{2,1,1}$, thus, according to (GALLIER, 2011), $\exp\left(\mathbf{A}^*\right)=\exp\left(\mathbf{A}\right)^*$, and it follows that

$$\mathbf{y_p}=\sum_{j=1}^{n}\sum_{k=0}^{n_k}c_{jk}\mathscr{H}\left(t-t_{jk}\right)\left[\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)-\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)^*\right]\left[\mathbf{C}-2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{e}_j, \quad (367)$$

which further simplifies to

$$\mathbf{y_p}=2i\sum_{j=1}^{n}\sum_{k=0}^{n_k}c_{jk}\mathscr{H}\left(t-t_{jk}\right)\Im\left(\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)\right)\left[\mathbf{C}-2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{e}_j. \quad (368)$$

Let Equation (367) be expanded as

$$\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=0}^{n_k} c_{jk}\mathscr{H}\left(t-t_{jk}\right)\left[\underbrace{\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{M}^{-1}\mathbf{e}_j}_{\boldsymbol{\chi}_1} - \underbrace{\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)^{*}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{M}^{-1}\mathbf{e}_j}_{\boldsymbol{\chi}_2}\right]. \tag{369}$$

Using Eq. (252) and evaluating the complex-conjugate of $\boldsymbol{\chi}_1$,

$$\begin{aligned}\boldsymbol{\chi}_1^{*} &= \exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)^{*}\left[\left(-2i\Im\left(\mathbf{F}_{2,1,1}\right)\right)^{*}\right]^{-1}\mathbf{M}^{-1}\mathbf{e}_j \\ &= \exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)^{*}\left[2i\Im\left(\mathbf{F}_{2,1,1}\right)\right]^{-1}\mathbf{M}^{-1}\mathbf{e}_j \\ &= -\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)^{*}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{M}^{-1}\mathbf{e}_j = -\boldsymbol{\chi}_2,\end{aligned} \tag{370}$$

thus, Equation (369) can be simplified to

$$\begin{aligned}\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=0}^{n_k} c_{jk}\mathscr{H}\left(t-t_{jk}\right)&\left[\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{M}^{-1}\mathbf{e}_j\right. \\ &\left.+\left(\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{M}^{-1}\mathbf{e}_j\right)^{*}\right],\end{aligned} \tag{371}$$

which further simplifies to

$$\mathbf{y_p} = 2\sum_{j=1}^{n}\sum_{k=0}^{n_k} c_{jk}\mathscr{H}\left(t-t_{jk}\right)\Re\left(\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t\right)\right)\left[\mathbf{C}-2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{e}_j\right). \tag{372}$$

Example

Dirac's delta is an important distribution in engineering and simulation since it can be used to model impact and peaks in excitation. Therefore, it is widely used in mechanical engineering to model transient vibration due to impact and in electrical engineering to model peaks in current or tension. These phenomena are present in almost any structure and circuit, although obtaining an accurate response is cumbersome and computationally expensive, due to the necessity of interpolating the Dirac's delta and due to the extremely tiny time step. Thus, this example shows the ease of obtaining an analytical solution that does not need any interpolation of the Dirac's delta and that is quite simple to evaluate. Hence, consider the 3 DOF problem with $\beta = 1\times 10^{-2}$ and excitation

$$\mathbf{f} = \begin{Bmatrix} 0 \\ \delta\left(t-1\right)-\delta\left(t-5\right) \\ 0 \end{Bmatrix}, \tag{373}$$

such that $g_2(t) = c_{20}\delta(t-t_0) + c_{21}\delta(t-t_1)$ with $c_{20} = 1$, $c_{21} = -1$, $t_0 = 1$ and $t_0 = 5$. Using the conjugacy of $\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,1}$, one can use Eq. (368) directly,

$$\mathbf{y_p} = 2i\left(\mathscr{H}(t-1)\,\Im\left(\exp\left(\mathbf{F}_{2,1,1}(1-t)\right)\right) - \mathscr{H}(t-5)\,\Im\left(\exp\left(\mathbf{F}_{2,1,1}(5-t)\right)\right)\right) \left[\mathbf{C} - 2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{e}_2,$$

and as shown, for homogeneous initial conditions and impulse excitation, $\mathbf{y} = \mathbf{y_p}$.

The solution using State Variables, given by Eq, (277), are particularized for this example as

$$\mathbf{y}_{SV}(t) = \left[\exp\left(\bar{\mathbf{D}}(1-t)\right)\mathscr{H}(t-1) - \exp\left(\bar{\mathbf{D}}(5-t)\right)\mathscr{H}(t-5)\right]\bar{\mathbf{B}}\mathbf{e}_2 + \exp\left(-\bar{\mathbf{D}}t\right)\mathbf{C}_{1,SV}, \quad (374)$$

where $\mathbf{C}_{1,SV}$ is a vector of integration constants associated to the initial conditions. As both solutions depend upon matrix exponentials only, Eq. (372) and Eq. (374), it is straightforward to conclude that the cost of evaluation of such exponential maps become much larger when the dimensionality is increased (HIGHAM, 2008; MOLER; LOAN, 2003). This solution is compared to the solution given by the generalized integrating factor in Fig. 12. The response was also compared to the solution obtained by using the traditional Newmark-beta method with $\Delta t = 0.001$. Each impulse was approximated by (EFTEKHARI, 2015)

$$\delta(t-t_0) \approx \frac{1}{2\varepsilon}\left(1 + \cos\left(\frac{\pi(t-t_0)}{\varepsilon}\right)\right) \quad t_0 - \varepsilon \leq t \leq t_0 + \varepsilon, \quad (375)$$

with $\varepsilon = \Delta t$ to impose the δs to the numerical method (the proposed methodology and State Variables do not need such approximation). Solutions are shown in Fig. 12 where the solid lines correspond to the real part of the proposed approach, the dotted dark lines to the solution obtained by using State Variables and the dotted red lines to the numerical solution (Newmark-beta method with $\Delta t = 0.001$s).

3.4.4 Heaviside step function

Let the normalized excitation vector be defined as

$$\bar{\mathbf{f}} = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1 + \ldots + g_n(t)\mathbf{v}_n, \quad (376)$$

with

$$g_j(t) = \sum_{k=0}^{n_k} f_{jk}(t)\mathscr{H}\left(t - t_{jk}\right), \quad (377)$$

where $f_{jk}(t)$ are functions of t, n_k the number of terms and $t_{jk} \in \mathbb{R}$ time shifts. Again, $\mathscr{H}$ is the Heaviside step function defined in Eq. (120).

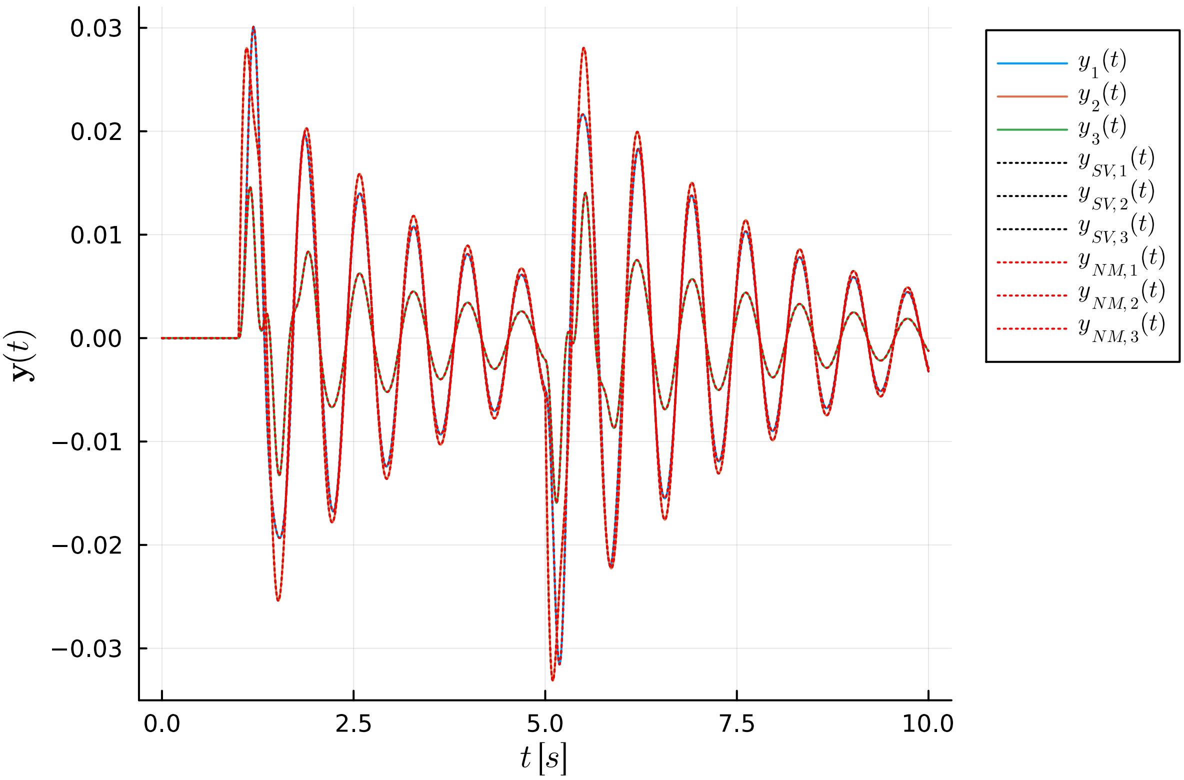

Figure 12 – Complete solution for the three DOFs example subjected to two opposed Dirac's deltas at $t = 1$ and $t = 5$, respectively. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $y_{SV,1}$, $y_{SV,2}$ and $y_{SV,3}$, obtained by using State Variables, are shown as black dotted lines. Solutions $y_{NM,1}$, $y_{NM,2}$ and $y_{NM,3}$, obtained by using the Newmark-beta method, are shown as red dotted lines.

Substituting in Eq. (442), yields

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} f_{jk}(t) \mathscr{H}\left(t - t_{jk}\right) \mathbf{v}_j \, \mathrm{d}t \, \mathrm{d}t, \tag{378}$$

using the linearity of the integral operator and as $\mathbf{v}_j$ does not depend in time results in

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) f_{jk}(t) \mathscr{H}\left(t - t_{jk}\right) \mathrm{d}t \, \mathrm{d}t \, \mathbf{v}_j. \tag{379}$$

Equation (360) is used to evaluate the inner convolution,

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) f_{jk}(t) \, \mathrm{d}t \; \mathscr{H}\left(t - t_{jk}\right) \mathrm{d}t \, \mathbf{v}_j, \tag{380}$$

and, again, Equation (360) is used to evaluate the outer integral,

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \mathscr{H}\left(t - t_{jk}\right) \int_{t_{jk}}^{t} \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) f_{jk}(t)\,\mathrm{d}t\,\mathrm{d}t\,\mathbf{v}_j. \tag{381}$$

This solution depends on the function that multiplies the Heaviside function and the convolution can be trivially evaluated when $\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}$ is non-singular, since the convolution is evaluated for times t_{jk} different from t_0. Nonetheless, for the singular case, one can evaluate the convolution by using the Jordan canonical form (HIGHAM, 2008).

The cases derived in previous subsections cover an extent set of widely used excitation functions, *e.g.*, harmonic, polynomial and constant (a constant function is a polynomial of order 0). Thus, the integrals in Eq. 447 can be found by using previous results as reference. In the following, the result for a second order polynomial is presented.

3.4.4.1 Particularizing Heaviside excitation for second order polynomial

Let $f_{j,k}$ in Eq. (377) to be particularized to a second order polynomial, *i.e.*,

$$g_j(t) = \sum_{k=0}^{n_k} \left(c_{jk0} + c_{jk1} t + c_{jk2} t^2\right) \mathscr{H}\left(t - t_{jk}\right). \tag{382}$$

Applying this excitation in Eq. (447) yields

$$\mathbf{y_p}^{(2)} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \mathscr{H}\left(t - t_{jk}\right) \int_{t_{jk}}^{t} \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \left(\int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) c_{jk0}\,\mathrm{d}t + \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) c_{jk1} t\,\mathrm{d}t + \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) c_{jk2} t^2\,\mathrm{d}t\right) \mathrm{d}t\,\mathbf{v}_j. \tag{383}$$

The inner convolutions are analytically integrated using Eq. (339), and, rearranging the terms, one gets

$$\mathbf{y_p}^{(2)} = \exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t\right)\sum_{j=1}^{n}\sum_{k=0}^{n_k}\mathscr{H}\left(t-t_{jk}\right)\int_{t_{jk}}^{t} c_{jk2}t^2\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\mathbf{F}_{2,1,1}^{-1}+ \\ t\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left(2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\right. \\ \left.-c_{jk1}\mathbf{F}_{2,1,1}^{-2}+c_{jk0}\mathbf{F}_{2,1,1}^{-1}\right)+\exp\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]t\right)\left(-c_{jk2}t_{jk}^2\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}+\right. \\ 2c_{jk2}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2}-c_{jk1}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1} \\ -2c_{jk2}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-3}+c_{jk1}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2} \\ \left.-c_{jk0}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}\right)\mathrm{d}t\,\mathbf{v}_j, \tag{384}$$

which can be again convoluted by using Eq. (339). Multiplying by $\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t\right)$ yields

$$\mathbf{y_p}^{(2)} = \sum_{j=1}^{n}\sum_{k=0}^{n_k}\mathscr{H}\left(t-t_{jk}\right)\left(c_{jk2}t^2\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\mathbf{F}_{2,1,1}^{-1}+t\left(\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\right(\right. \\ \left.\left.c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)-2c_{jk2}\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-2}\mathbf{F}_{2,1,1}^{-1}\right)+2c_{jk2}\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-3}\mathbf{F}_{2,1,1}^{-1} \\ -\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-2}\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)+\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\left(2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\right. \\ \left.-c_{jk1}\mathbf{F}_{2,1,1}^{-2}+c_{jk0}\mathbf{F}_{2,1,1}^{-1}\right)+\exp\left(-\mathbf{F}_{2,1,1}t\right)\left(\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right)^{-1}(\\ -c_{jk2}t_{jk}^2\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}+2c_{jk2}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2} \\ -c_{jk1}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-3} \\ \left.+c_{jk1}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2}-c_{jk0}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}\right)+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{jk}-t\right)\right)(\\ -c_{jk2}t_{jk}^2\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\mathbf{F}_{2,1,1}^{-1}+2c_{jk2}t_{jk}\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-2}\mathbf{F}_{2,1,1}^{-1} \\ -2c_{jk2}\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-3}\mathbf{F}_{2,1,1}^{-1}-t_{jk}\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)+\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-2}(\\ \left.c_{jk1}\mathbf{F}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)-\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\left(2c_{jk2}\mathbf{F}^{-3}-c_{jk1}\mathbf{F}^{-2}+c_{jk0}\mathbf{F}^{-1}\right)\Big)- \\ \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{jk}-t\right)-\mathbf{F}_{2,1,1}t_{jk}\right)\left(\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right)^{-1}(\\ -c_{jk2}t_{jk}^2\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}+2c_{jk2}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2}- \\ c_{jk1}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-3}+ \\ \left.\left.c_{jk1}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2}-c_{jk0}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}\right)\right)\mathbf{v}_j. \tag{385}$$

By using Eqs. (651) and (644), the term $\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)$ can be commuted with $\left(\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right)^{-1}$. Thus, the expression can be further simplified to

$$\mathbf{y_p}^{(2)} = \sum_{j=1}^{n}\sum_{k=0}^{n_k} \mathscr{H}\left(t-t_{jk}\right)\Big\{c_{jk2}t^2\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+t\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}$$
$$\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)-2c_{jk2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\Big)+2c_{jk2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-3}\mathbf{F}_{2,1,1}^{-1}$$
$$-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big(2c_{jk2}\mathbf{F}_{2,1,1}^{-3}$$
$$-c_{jk1}\mathbf{F}_{2,1,1}^{-2}+c_{jk0}\mathbf{F}_{2,1,1}^{-1}\Big)+\exp\left(\mathbf{F}_{2,1,1}\left(t_{jk}-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}$$
$$\Big(-\left(c_{jk2}t_{jk}^2+c_{jk1}t_{jk}+c_{jk0}\right)\mathbf{F}_{2,1,1}^{-1}+\left(2c_{jk2}t_{jk}+c_{jk1}\right)\mathbf{F}_{2,1,1}^{-2}-2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\Big)$$
$$+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{jk}-t\right)\right)\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big(\mathbf{F}_{2,1,1}^{-1}\left(-c_{jk2}t_{jk}^2-c_{jk1}t_{jk}-c_{jk0}\right)$$
$$+\mathbf{F}_{2,1,1}^{-2}\left(2c_{jk2}t_{jk}+c_{jk1}\right)-2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\Big)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\Big(\mathbf{F}_{2,1,1}^{-1}\left(2c_{jk2}t_{jk}+c_{jk1}\right)$$
$$-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\Big)-2c_{jk2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-3}\mathbf{F}_{2,1,1}^{-1}-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\Big(-\mathbf{F}_{2,1,1}^{-1}\Big(c_{jk2}t_{jk}^2$$
$$+c_{jk1}t_{jk}+c_{jk0}\Big)+\mathbf{F}_{2,1,1}^{-2}\left(2c_{jk2}t_{jk}+c_{jk1}\right)-2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\Big)\Big)\Big\}\mathbf{M}^{-1}\mathbf{e}_j. \quad (386)$$

Heaviside step function are used to model abruptly shifting forces and electrical currents. This excitation type is extremely important in mechanical vibration analysis, circuitry analysis and signal processing. Thus, enabling the exact modelling of a system subject to such discontinuous excitation is paramount to forecast how it will respond to it and, then, a lot of procedures may be taken to adequate it within desired behavior, like optimization. Hence, two examples are provided of excitation functions that can be parameterized using Heaviside steps multiplied by polynomials.

Example

Consider the 3 DOF problem with $\beta = 1\times10^{-2}$ and excitation

$$f_2(t) = \begin{cases} -30+40t-10t^2 & 1\leq t\leq 3 \\ 0 & t\in[0,1)\cup(3,\infty] \end{cases} \quad (387)$$

or

$$\mathbf{f} = \left\{\begin{array}{c} 0 \\ (-30+40t-10t^2)\mathscr{H}\left(t-1\right)+(30-40t+10t^2)\mathscr{H}\left(t-3\right) \\ 0 \end{array}\right\}, \quad (388)$$

such that $c_{200} = -30$, $c_{201} = 40$, $c_{202} = -10$, $t_{20} = 1$, $c_{210} = 30$, $c_{211} = -40$, $c_{212} = 10$ and $t_{21} = 3$.

The solution using State Variables, given by Eq, (277), is particularized for this example as

$$\mathbf{y}_{SV}(t) = \left[\left[-10\bar{\mathbf{D}}^{-1}t^2 + t\left[40\bar{\mathbf{D}}^{-1} + 20\bar{\mathbf{D}}^{-2}\right] - 30\bar{\mathbf{D}}^{-1} - 40\bar{\mathbf{D}}^{-2} - 20\bar{\mathbf{D}}^{-3}\right.\right.$$
$$\left. + \exp\left(\bar{\mathbf{D}}(1-t)\right)\left[20\bar{\mathbf{D}}^{-2} + 20\bar{\mathbf{D}}^{-3}\right]\right]\mathscr{H}(t-1) + \left[10\bar{\mathbf{D}}^{-1}t^2 - t\left[40\bar{\mathbf{D}}^{-1} + 20\bar{\mathbf{D}}^{-2}\right]\right.$$
$$\left.\left. + 30\bar{\mathbf{D}}^{-1} + 40\bar{\mathbf{D}}^{-2} + 20\bar{\mathbf{D}}^{-3} + \exp\left(\bar{\mathbf{D}}(5-t)\right)\left[20\bar{\mathbf{D}}^{-2} - 20\bar{\mathbf{D}}^{-3}\right]\right]\mathscr{H}(t-5)\right]\bar{\mathbf{B}}\mathbf{e}_2$$
$$+ \exp\left(-\bar{\mathbf{D}}t\right)\mathbf{C}_{1,SV}, \quad (389)$$

where $\mathbf{C}_{1,SV}$ is a vector of integration constants associated to the initial conditions. As both solutions depend upon costly matrix operations, such as exponential maps and matrix inverses, it is straightforward to conclude that the evaluation of these operations become costlier when the dimensionality is increased. Also, as the State Variables doubles the dimensionality of the problem, one can expect much larger execution times for this approach. This solution is compared to the solution given by the generalized integrating factor in Fig. 13.

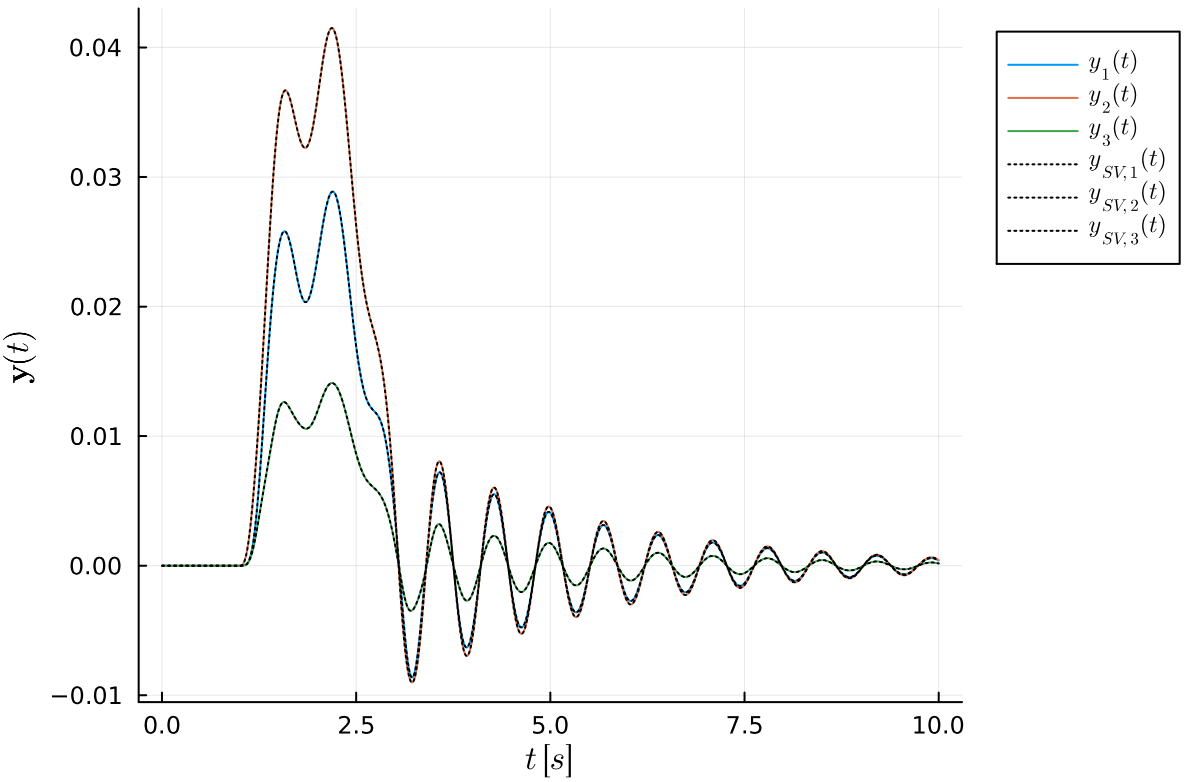

Figure 13 – Complete solution for the three DOFs example subjected to a quadratic load between $t = 1$ and $t = 3$s. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $y_{SV,1}$, $y_{SV,2}$ and $y_{SV,3}$ obtained by using State Variables are shown as black dotted lines.

Example

Consider the 3 DOF problem with $\beta = 1 \times 10^{-2}$. Assuming an unitary step at DOF 2 between $t=1$ and $t=5$

$$\mathbf{f}=\left\{\begin{array}{c}0\\ \mathscr{H}\left(t-1\right)-\mathscr{H}\left(t-5\right)\\ 0\end{array}\right\}, \tag{390}$$

such that $g_2(t)=c_{200}\mathscr{H}(t-t_{20})+c_{210}\mathscr{H}(t-t_{21})$ with $c_{200}=1$, $c_{210}=-1$, $t_{20}=1$ and $t_{21}=5$. Solution provided by Eq. (386) can be particularized to order zero

$$\begin{aligned}\mathbf{y_p}^{(0)}=\sum_{j=1}^{n}\sum_{k=0}^{n_k}\mathscr{H}\left(t-t_{jk}\right)\Big\{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}c_{jk0}\mathbf{F}_{2,1,1}^{-1}\\ -c_{jk0}\exp\left(\mathbf{F}_{2,1,1}\left(t_{jk}-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{jk}-t\right)\right)\\ \left(-c_{jk0}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+c_{jk0}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\right)\Big\}\mathbf{M}^{-1}\mathbf{e}_j.\end{aligned} \tag{391}$$

Using the data for this example

$$\begin{aligned}\mathbf{y_p}^{(0)}=\mathscr{H}\left(t-1\right)\Big\{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\\ -\exp\left(\mathbf{F}_{2,1,1}\left(1-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(1-t\right)\right)\\ \left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\right)\Big\}\mathbf{M}^{-1}\mathbf{e}_2+\\ \mathscr{H}\left(t-5\right)\Big\{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\\ +\exp\left(\mathbf{F}_{2,1,1}\left(5-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(5-t\right)\right)\\ \left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\right)\Big\}\mathbf{M}^{-1}\mathbf{e}_2,\end{aligned} \tag{392}$$

which is further simplified to

$$\begin{aligned}\mathbf{y_p}^{(0)}=\mathscr{H}\left(t-1\right)\Big\{\left[\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right]^{-1}\\ -\exp\left(\mathbf{F}_{2,1,1}\left(1-t\right)\right)\left[\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right]^{-1}+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(1-t\right)\right)\\ \left(-\left[\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right]^{-1}+\left[\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right]^{-1}\right)\Big\}\mathbf{e}_2+\\ \mathscr{H}\left(t-5\right)\Big\{\left[\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right]^{-1}\\ +\exp\left(\mathbf{F}_{2,1,1}\left(5-t\right)\right)\left[\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right]^{-1}+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(5-t\right)\right)\\ \left(\left[\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right]^{-1}-\left[\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right]^{-1}\right)\Big\}\mathbf{e}_2,\end{aligned} \tag{393}$$

and, by using Eq. (207), it reduces to

$$\mathbf{y_p}^{(0)} = \mathscr{H}(t-1)\Big\{\mathbf{K}^{-1} - \exp\left(\mathbf{F}_{2,1,1}(1-t)\right)\left[\mathbf{K}-\mathbf{M}\mathbf{F}_{2,1,1}^2\right]^{-1} - \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](1-t)\right)\Big(\mathbf{K}^{-1} - \left[\mathbf{K}-\mathbf{M}\mathbf{F}_{2,1,1}^2\right]^{-1}\Big)\Big\}\mathbf{e}_2 + \mathscr{H}(t-5)\Big\{\mathbf{K}^{-1} + \exp\left(\mathbf{F}_{2,1,1}(5-t)\right)\left[\mathbf{K}-\mathbf{M}\mathbf{F}_{2,1,1}^2\right]^{-1} + \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](5-t)\right)\Big(\mathbf{K}^{-1} - \left[\mathbf{K}-\mathbf{M}\mathbf{F}_{2,1,1}^2\right]^{-1}\Big)\Big\}\mathbf{e}_2, \tag{394}$$

The solution using the method of state variables, given by Eq, (277), is particularized for this example as

$$\mathbf{y}_{SV}(t) = \left[\left[\mathbf{I}-\exp\left(\bar{\mathbf{D}}(1-t)\right)\right]\mathscr{H}(t-1) - \left[\mathbf{I}-\exp\left(\bar{\mathbf{D}}(5-t)\right)\right]\mathscr{H}(t-5)\right]\bar{\mathbf{B}}\mathbf{e}_2 + \exp\left(-\bar{\mathbf{D}}t\right)\mathbf{C}_{1,SV}, \tag{395}$$

where $\mathbf{C}_{1,SV}$ is a vector of integration constants associated to the initial conditions. As both solutions depend upon costly matrix operations, such as exponential maps and matrix inverses, it is straightforward to conclude that the evaluation of these operations become costlier when the dimensionality is increased by a factor of 2, as when using State Variables. This remark is reassured as the inverses in Eq. (394) depend mostly on the stiffness matrix, $\mathbf{K}$, which is a sparse positive-definite matrix (ZIENKIEWICZ; TAYLOR; ZHU, 2013), thus, enabling the use of fast algorithms for the solution of the linear systems for each input vector, $\mathbf{e}_j$. This solution is compared to the solution given by the real part of the generalized integrating factor in Fig. 14.

3.4.4.2 Initial conditions of Heaviside and Dirac's delta excitation

As all previous excitation functions were split in the canonical basis of the $\mathbb{R}^n$, each integration was evaluated separately for each functional component and, then, multiplied by the respective basis vector. Nonetheless, for both the Heaviside and the Dirac's delta excitations, the component of the solution that multiply each basis vector has null response and derivative at the initial point. Consequently, the initial conditions for these excitations are zero such that $\mathbf{y_p}(t) = \mathbf{0}\ \forall t$, as discussed in Chapter 2. This result was observed in previous examples. The consideration of non homogeneous initial conditions is discussed in Appendix B.1.

3.4.4.3 Matrix complexity for polynomial particularized Heaviside

Heaviside excitation particularized to polynomial f_{jk} yields expensive matrices evaluations, like power and inverse operations. This is expected, since similar complexity is observed when addressing polynomial excitations. As it can be observed in Eq. (386), this complexity increases with the order of f_{jk}.

However, some calculation can be avoided by using the quadratic matrix equation, Eq. (208). For example, by factoring Eq. (208),

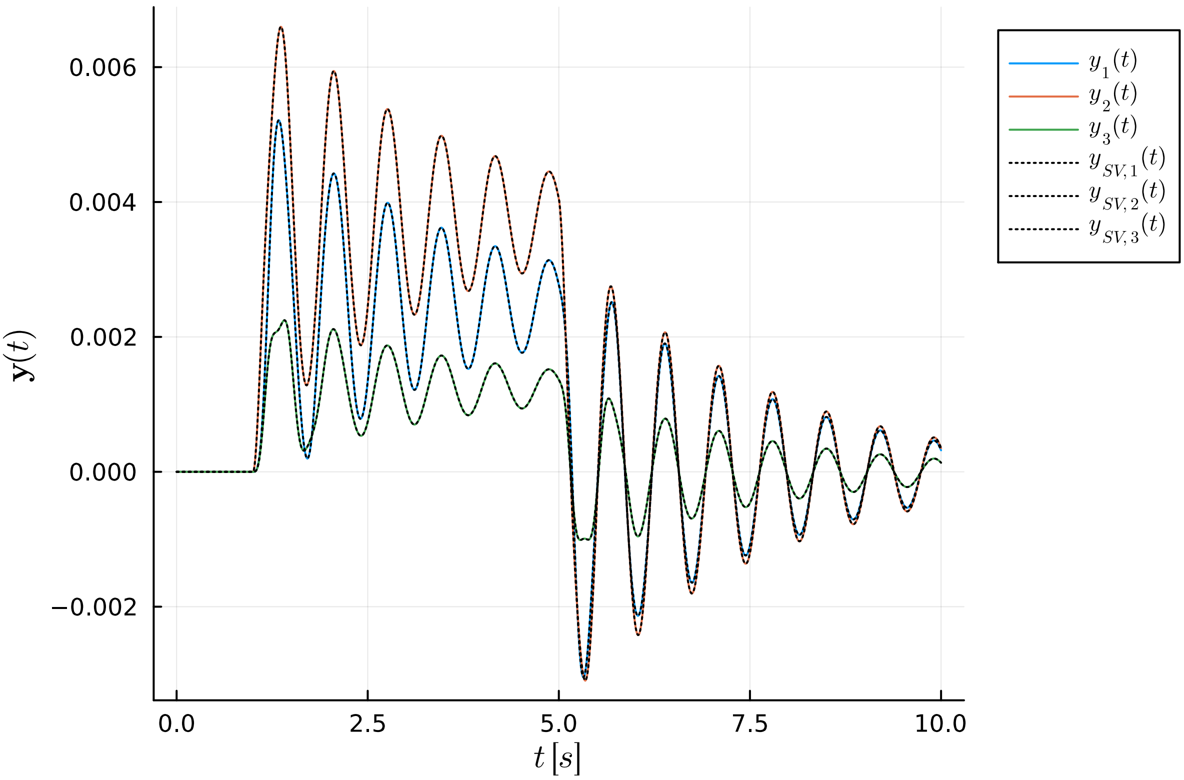

Figure 14 – Complete solution for the three DOFs example subjected to an unitary step between $t = 1$ and $t = 5$s. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $y_{SV,1}$, $y_{SV,2}$ and $y_{SV,3}$ obtained by using State Variables are shown as black dotted lines.

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{F}_{2,1,1}\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] + \bar{\mathbf{K}} = \mathbf{0}, \tag{396}$$

which is then rearranged to

$$\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] = \bar{\mathbf{K}}. \tag{397}$$

Taking the inverse of both sides, results in

$$\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1} = \bar{\mathbf{K}}^{-1} = \mathbf{K}^{-1}\mathbf{M}, \tag{398}$$

thus, the inverse of $\mathbf{F}_{2,1,1}$ can be calculated by

$$\mathbf{F}_{2,1,1}^{-1} = \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\mathbf{K}^{-1}\mathbf{M}; \tag{399}$$

conversely, the inverse of $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ can be evaluated by

$$\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} = \mathbf{K}^{-1}\mathbf{M}\mathbf{F}_{2,1,1}. \tag{400}$$

Hence, the two most common inverses in Eq. (386) are evaluated using just one inverse, the inverse of $\mathbf{K}$. These relations can also be used to decrease the computational cost for polynomial excitation. The inverse of $\mathbf{K}$ is particularly interesting in the Finite Element Analysis (FEA) context, since the stiffness matrix is sparse, enabling the use of fast, efficient and tailored algorithms for this kind of matrices (LI et al., 2008; BOLLHöFER; SCHENK; VERBOSIO, 2021). Therefore, the inverse of more complicate matrices, like $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$, can be computed by using an inverse that is cheaper to evaluate. It is important to stress that matrices are not usually inverted in the numerical implementation, but $\mathbf{K}$ is the coefficient matrix used to solve a linear system of equations. Nonetheless, all the previous discussion still applies., since dealing directly with $\mathbf{K}$ is much cheaper than dealing with the original forms.

The matrix $\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]$ is trickier to simplify, but it can be done using Eq. (239),

$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] = -\left(\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right)^{\frac{1}{2}}, \tag{401}$$

and if $\bar{\mathbf{C}}$ is given by proportional damping,

$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] = -\left(\beta^2\bar{\mathbf{K}}^2 + (2\alpha\beta - 4)\,\bar{\mathbf{K}} + \alpha^2\mathbf{I}\right)^{\frac{1}{2}}. \tag{402}$$

Equation (402) can be approximated if one assumes both α and β as very small values. In such case

$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] \approx -2i\bar{\mathbf{K}}^{\frac{1}{2}}, \tag{403}$$

and the inverse of $\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]$ can be approximated to

$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \approx \frac{i}{2}\left(\mathbf{M}^{-1}\mathbf{K}\right)^{-\frac{1}{2}} = \frac{i}{2}\left(\mathbf{K}^{-1}\mathbf{M}\right)^{\frac{1}{2}}. \tag{404}$$

Again, a matrix inverse can be calculated using the inverse of the stiffness matrix. The approximation presented in Eq. (403), however, can also be used in Eq. (239) to reduce computation costs. Such approximations were not used in the examples discussed in this text.

3.5 RUNNING TIME COMPARISON

The proposed method was particularized for four families of excitation functions that are common in real-world problems, such as exponential/periodic, polynomial, Dirac's delta and

Heaviside step functions. For all of the particularizations, at least one practical example was provided and compared to the analytical solution obtained by State Variables. Exceptionally, for excitation due to Dirac's delta, the Newmark-beta numerical method was also used due to its extensive usage in the literature and to illustrate how such excitation is numerically parameterized. Regarding the analytical comparison with State Variables, it was observed that the response due to the latter induced higher computational cost, since costly operations over matrix of doubled dimension.

In the case of Newmark-beta and other numerical methods, it is important to highlight the need for a finer time discretization, to reduce the propagation of errors. Despite that, it is also important to stress that in the implementation of the Newmark-beta, (HUGHES, 2000; LINDFIELD; PENNY, 2019), there are at least 3 matrix-vector multiplications per time step ($\mathscr{O}\left(n^2\right)$, (JIN; CHEW, 2005), where n is the dimensionality of the problem) and the solution of a linear system. This linear system can be pre-factorized such that a forward and a backward substitutions are needed at each time step (($\mathscr{O}\left(n^2\right)$ (FORD, 2015)). Hence, it is estimated that each iteration of the Newmark-beta method has a complexity of $5\mathscr{O}\left(n^2\right)$.

If the scheme presented in Eq. (650) is used to evaluate the homogeneous analytical response in a set of uniformly distributed time points, the exponential map must be evaluated only once at a pre-processing step. Hence, a matrix-vector multiplication of complexity $\mathscr{O}\left((2n)^2\right)$ is needed at each time step when using State Variables. The proposed approach requires two matrix-vector multiplications when the problem is over damped, with complexity $\mathscr{O}\left(n^2\right)$ each, or just a single matrix-vector product of complexity $\mathscr{O}\left(n^2\right)$ if the system is under damped.

Henceforth, regardless of time discretization (which is expected to be fine in numerical methods), the Newmark-beta is already expected to have more floating point operations per time step, that build up as more points are added to the time span.

The example used to assess the proposed approach with Dirac's delta was used as a starting point to devise a numerical experiment. A FEM model with increasing number of DOFs is subjected to the same loading (two Dirac's deltas) and to the same settings: time span ($t \in [0, 10]\,s$), time discretization ($\Delta t = 0.001\ s$) and the damping coefficient ($\beta = 10^{-6}$).

The results are shown in Fig. 15. The blue line corresponds to the execution time for the Newmark-beta method, whereas the orange line corresponds to the execution time of State Variables and the green line to the execution time of the proposed approach. It is straightforward to observe that the execution time of the Newmark-beta method rises much quicker than those of the two analytical methods. It shows the advantage of using analytical techniques for linear problems despite the preference for numerical approaches in the literature. Beyond that, it is clear that the proposed approach spared computational time by a significant margin, that is visible in Fig. 15. Thereby, the proposed approach can be also computationally efficient and a viable option for simulation of linear systems with higher dimensionality.

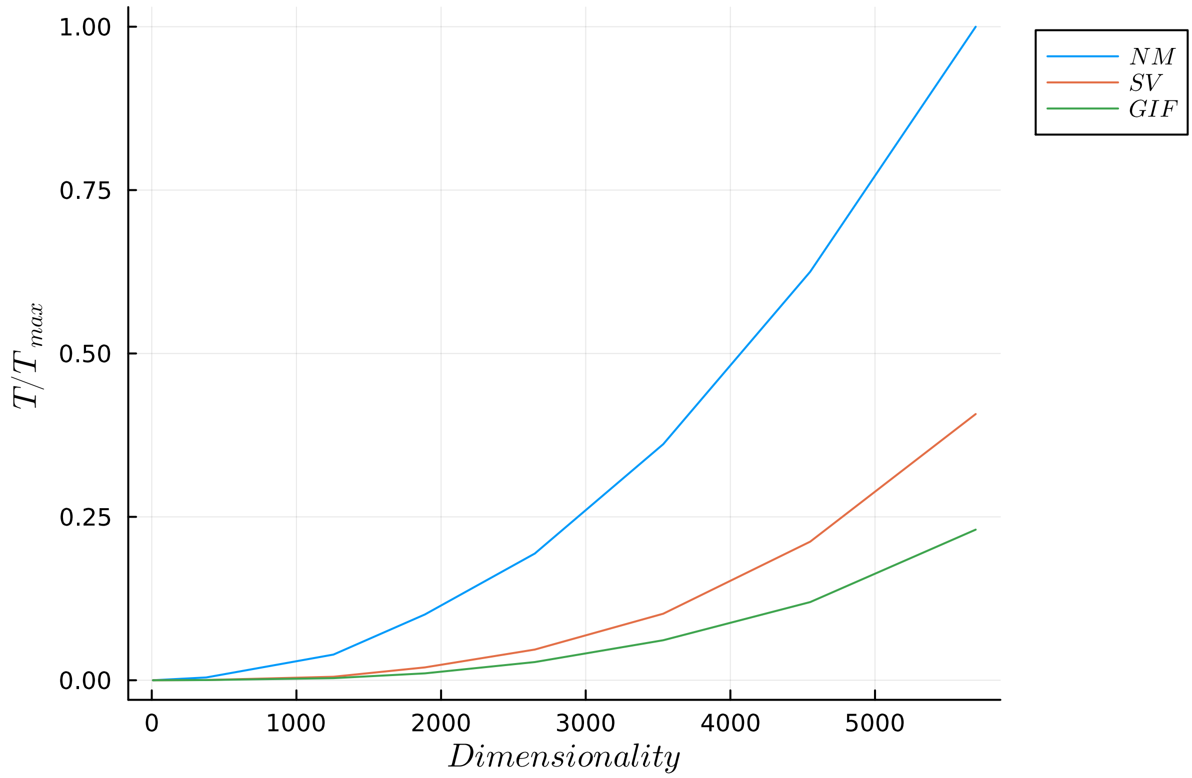

Figure 15 – Computation time to evaluate the response due to Dirac's excitation as a function of the dimensionality of the problem. The complete response was calculated using the Newmark-beta method (*NM*, blue line), State Variables (*SV*, orange line) and the proposed approach (*GIF*, green line). T is the computation time at a given dimensionality, while T_{max} is the maximum time measured for all methods in the entire experiment.

3.6 APPLICATION IN INTEGRAL MEASURES

Integral measures arise in many engineering applications, especially in optimization of structures subject to dynamic loads. In (JOG, 2002), a review of the integral measures is provided, while, in (ZHU; ZHANG; BECKERS, 2009), examples with topology optimization are given. One of such integral measures is power input, which can be used to mitigate vibrations in structures using topology optimization (SILVA; NEVES; LENZI, 2019; SILVA; NEVES; LENZI, 2020). Calculate such measures analytically and, thus, having analytical expressions for their sensitivity analysis is paramount. Hence, the generalized integrating factor can be used both for determining system response and for evaluating objective or constraint functions dependent upon integral measures.

3.6.1 Input energy with periodic excitation

The input energy along a time span can be calculated as

$$E = \int_{t_0}^{t_f} \mathbf{f}^T \dot{\mathbf{y}}\, \mathrm{d}t, \tag{405}$$

where T stands for the transpose of the excitation vector. If the excitation function is periodic, Equations (282) and (306) and the derivative of Eq. (223) can be directly substituted in Eq. (405),

$$\begin{aligned} E = \sum_{j=1}^{n}\sum_{k=1}^{n_k}\sum_{l=1}^{n}\sum_{m=1}^{n_k} \int_{t_0}^{t_f} & c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \mathbf{e}_j^T c_{lm} \beta_{lm} \exp\left(\beta_{lm} t + \phi_{lm}\right) \mathbf{k}_{lm} + \\ & c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \mathbf{e}_j^T \left[-\exp\left(-\mathbf{F}_{2,1,1} t\right) \mathbf{F}_{2,1,1} \mathbf{C}_2 + \right. \\ & \left. \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] \mathbf{C}_1\right] \mathrm{d}t. \end{aligned} \tag{406}$$

The integral above has two parts - one corresponding to the particular solution with no matrix exponential and one corresponding to the homogeneous solution, that has indeed matrix exponentials. The part due to the particular response has a straightforward solution, while the one due to the homogeneous response has a solution using Eq. (289), thus,

$$\begin{aligned} E = \sum_{j=1}^{n}\sum_{k=1}^{n_k}\sum_{l=1}^{n}\sum_{m=1}^{n_k} & \frac{c_{jk} c_{lm} \beta_{lm}}{\beta_{jk} + \beta_{lm}} \exp\left(\left(\beta_{jk} + \beta_{lm}\right) t + \phi_{jk} + \phi_{lm}\right)\Big|_{t_0}^{t_f} \mathbf{e}_j^T \mathbf{k}_{lm} + \\ & c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \mathbf{e}_j^T \left[\exp\left(-\mathbf{F}_{2,1,1} t\right) \left[\mathbf{F}_{2,1,1} - \beta_{jk} \mathbf{I}\right]^{-1} \mathbf{F}_{2,1,1} \mathbf{C}_2 + \right. \\ & \left. \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}} + \beta_{jk} \mathbf{I}\right]^{-1} \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] \mathbf{C}_1\right]\Big|_{t_0}^{t_f}. \end{aligned} \tag{407}$$

Both matrix inverses in Eq. (407) are the result of integrating the homogeneous parcel of the solution, therefore, this part will be common to the input energy calculation of all responses due to continuous excitation. These inverses can be avoided if the homogeneous response is neglected altogether, like in (SILVA; NEVES; LENZI, 2019; SILVA; NEVES; LENZI, 2020). Nevertheless, the complexity of all these matrix operations is dependent upon the dimensionality of the problem only and totally independent of the time span, in contrast to numerical methods, whose evaluation points would have to be again numerically integrated. In consequence, as the length of the time span increases, the approach using the generalized integrating factor gets computationally more efficient, besides its intrinsic accuracy.

3.7 OPEN-SOURCE REPOSITORY - GIFFNDOF

The source code of the computer implementation of the proposed formulation is available in the public repository <https://github.com/CodeLenz/Giffndof.jl>. The open source Julia language (BEZANSON et al., 2012) is used. Instructions for downloading and usage can be found in the documentation and in the examples provided in the GitHub page of the repository.

Two different implementations are available: a continuous one, where the complete, the particular and the homogeneous responses are given, for each tye of excitation, as functions of time and a discrete version where the complete response is computed for a pre-defined grid of time steps and for any combination of exponential, polynomial and Dirac's impulse excitations.

3.8 FINAL REMARKS OF THE CHAPTER

This chapter extended the use of the GIF to coupled systems of second order ODEs. Expressions for the time-dependent coefficient case were derived in general form. These forms were later particularized to the constant coefficient case where it was shown that under mild assumptions about coefficient matrix $\bar{\mathbf{C}}$ the integrating factor can be found in closed form. Analytical particular solutions were derived for different forms of continuous and discontinuous excitations. Complicate expressions for loading can be generated by using a linear combination of polynomials multiplied by Heavisides and also by combining the analytical solutions derived in this manuscript.

Examples showed that the proposed approach is accurate and can be made efficient, not suffering with common issues found in traditional numerical approaches, like stability associated to time discretization and interpolation errors. Initial conditions can be imposed at any given time t_0, not only in the extremes of the interval. Actually, no time span is needed to evaluate the response when using the proposed approach.

Implementation ideas regarding the exponential maps in the solutions were launched and tested, which made the computational implementation efficient. It was observed through numerical experiments that the GIF spent less computational effort than well-established techniques, such as Newmark-beta and State Variables methods.

Comparisons against other analytical approach, namely the Laplace transform, demonstrated that the latter is indeed impractical for large problems and cannot be scaled to n-dimensional problems like the GIF can. Summing these results up, it follows that the GIF is a practical option to solve systems of coupled ODEs, both reliably and with computational efficiency.

4 HEAVISIDE SERIES

In Chapter 3, the importance of solving systems of coupled linear ODEs was highlighted in many practical applications. For this reason, the Generalized Integrating Factor was extended to tackle these systems and particularized for some families of excitation functions. The advantages of the method, such as accuracy and lesser computational effort, were shown through numerical experiments.

The particular solutions in Chapter 3, however, depended on a double convolution. Hence, one might ask: *what if an excitation function does not have a analytical convolution?*While another might also ask: *what if I do not want to trouble myself to evaluate the convolution for an excitation function that interests me?* The answer is to approximate such cases of excitation function in a way that does have analytical convolution and that has already been derived.

From this intuition, a semi-analytical method is formulated in this chapter. The original excitation function is approximated using series of Heaviside step functions multiplied by polynomial coefficients. As this type of excitation had already been developed in Chapter 3, the solution to the approximated function will be analytic. For this reason, the method is semi-analytic, since the solution is analytic but the excitation is an approximation, and the accuracy is solely dependent on the quality of the approximation of the excitation function.

It will be shown in this chapter that the proposed semi-analytical method, called Heaviside Series method, keeps the advantage of lesser computational effort from the original GIF technique. It will be shown through numerical experiments that the method has an order of convergence between 2 and 4 and that it is, consequently, more accurate and faster than the Newmark-beta method. The numerical stability of the Heaviside Series method, HS for short, is also explored and it is shown to be linked to the eigenvalues of the matrices $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ - if they have positive real part.

The eigenvalues of these matrices show that the method is unconditionally stable when damping is *positive*. It is proven that, when the eigenvalues have negative real part, it is not the method that is unstable but the system as a whole. Consequently, the response is not artificially damped as in other numerical methods. Conditions for the damping being positive are investigated for Rayleigh proportional damping.

4.1 HEAVISIDE SERIES FOR FUNCTION APPROXIMATION

In many real-world and mathematical problems, functions must be approximated to fit some conditions. Among different uses, two classes of these problems are very common: data regression, when a function must be fit to data to infer and predict over unknown data and approximate solutions for differential equations, when a candidate solution is adjusted to fit the model's differential equation by minimizing a residue metric, such as in Galerkin method (ZIENKIEWICZ; TAYLOR; ZHU, 2013; GAUL; KÖGL; WAGNER, 2012). Hence, developing new ways of representing and approximating functions is indeed an important and current research topic.

4.1.1 One dimensional functions $f: V \to V$

Let a function $f(t),\ f: V \to V$, be approximated as a series of polynomials $\hat{c}_{k,m}t^m \in V$ up to power n_m multiplied by Heaviside step functions at n_k discrete time steps t_k

$$f(t) \approx \tilde{f}(t) = \sum_{k=0}^{n_k}\sum_{m=0}^{n_m} \hat{c}_{k,m}t^m \mathscr{H}(t-t_k), \tag{408}$$

where $\mathscr{H}$ is the Heaviside step function defined in Eq. (120).

As the function is essentially approximated by a polynomial of order n_m, it is fair to say that Eq. (408) provides an approximation of order n_m for $f(t)$, much like using Taylor series. Nonetheless, the coefficients of the polynomials are updated according to t, such that there is no need to center the approximation around a point like in Taylor series. Thus, the calculation of coefficients $\hat{c}_{k,m}$ is carried out differently.

The main idea for evaluating the coefficients $\hat{c}_{k,m}$ is to preserve the integral of the original function, $f(t)$ and its derivatives at each point t_k, depending on the order of approximation but regardless of domain discretization. Examples with zero, first and second order approximations are presented in the following and provide a glance at the niche of application of this series representation.

4.1.1.1 Zero order approximation

For zero order approximation, $n_m = 0$, Eq. (408) reduces to

$$\tilde{f}(t) = \sum_{k=0}^{n_k} \hat{c}_k \mathscr{H}(t-t_k). \tag{409}$$

Using the preservation of the integral of $f(t)$ at each interval $t \in [t_l, t_{l+1}]$ and the mean value theorem for integrals

$$\tilde{f}(t) = \sum_{k=0}^{l} \hat{c}_k = \frac{1}{\Delta t_l}\int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t,\ \ t_l \le t \le t_{l+1}, \tag{410}$$

where $\Delta t_l = t_{l+1} - t_l$ and $0 \le l \le n_k$. Thus, the l-th coefficient, $\hat{c}_l$, is evaluated by

$$\hat{c}_l = \frac{1}{\Delta t_l}\int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \sum_{k=0}^{l-1} \hat{c}_k. \tag{411}$$

4.1.1.2 First order approximation

For first order approximation, $n_m = 1$, Eq. (408) reduces to

$$\tilde{f} = \sum_{k=0}^{n_k} \left(\hat{c}_{k,0} + \hat{c}_{k,1} t\right) \mathscr{H}\left(t - t_k\right). \tag{412}$$

The approximation at interval $t \in [t_l, t_{l+1}]$ is given by

$$\tilde{f}(t) = \sum_{k=0}^{l} \hat{c}_{k,0} + \sum_{k=0}^{l} \hat{c}_{k,1} t = a_{0,l} + a_{1,l} t, \;\; t_l \leq t \leq t_{l+1}. \tag{413}$$

The slope of this polynomial, $a_{1,l}$, can be tailored to be equal to

$$a_{1,l} = \frac{f\left(t_{l+1}\right) - f\left(t_l\right)}{\Delta t_l}, \tag{414}$$

which, by the mean value theorem for derivatives, guarantees that the derivative of $f(t)$ coincides with the derivative of $\tilde{f}$ at one point within the given interval, at least. Coefficient $a_{0,l}$ is obtained by

$$\int_{t_l}^{t_{l+1}} \tilde{f}(t)\,\mathrm{d}t = a_{0,l}\Delta t_l + \frac{a_{1,l}}{2}\left(t_{l+1}^2 - t_l^2\right) = \int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t, \tag{415}$$

such that,

$$a_{0,l} = \frac{1}{\Delta t_l}\int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \frac{a_{1,l}}{2\Delta t_l}\left(t_{l+1}^2 - t_l^2\right). \tag{416}$$

Thus, using Eq. (413), the coefficients are evaluated as

$$\hat{c}_{l,0} = \frac{1}{\Delta t_l}\int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \frac{a_{1,l}}{2\Delta t_l}\left(t_{l+1}^2 - t_l^2\right) - \sum_{k=0}^{l-1} \hat{c}_{k,0} \tag{417}$$

and

$$\hat{c}_{l,1} = \frac{f\left(t_{l+1}\right) - f\left(t_l\right)}{\Delta t_l} - \sum_{k=0}^{l-1} \hat{c}_{k,1}. \tag{418}$$

This approximation guarantees that the integral of $\tilde{f}(t)$ is the same as the integral of $f(t)$. It also enforces that the mean value of the derivative of $f(t)$ and the derivative of $\tilde{f}(t)$ are equal in each time interval. Both properties are observed regardless of the time step used in the approximation.

4.1.1.3 Second order approximation

For second order approximation, $n_m = 2$, Eq. (408) reduces to

$$\tilde{f}(t) = \sum_{k=0}^{n_k} \left(\hat{c}_{k,0} + \hat{c}_{k,1}t + \hat{c}_{k,2}t^2\right) \mathscr{H}\left(t - t_k\right). \tag{419}$$

At interval $t \in [t_l, t_{l+1}]$ the approximation function is given by

$$\tilde{f}(t) = \sum_{k=0}^{l} \hat{c}_{k,0} + \sum_{k=0}^{l} \hat{c}_{k,1}t + \sum_{k=0}^{l} \hat{c}_{k,2}t^2 = a_{0,l} + a_{1,l}t + a_{2,l}t^2, \;\; t_l \le t \le t_{l+1}. \tag{420}$$

For this order of approximation, the first derivative of the HS representation is set to be equal to the derivative of $f(t)$ at the vicinity of each time point t_k, what gives the advantage that the first derivative is smooth, although not defined at points t_k. Thus, the derivative of the representation, $\tilde{f}$, is continuous in the domain $\Omega: \; (-\infty, \infty) - \{t_l\}, l \in \{0, 1, \ldots, n_k\}$. It is straightforward to observe that, if f is differentiable at t_l, t_l is an accumulation point and the limit of the derivative of $\tilde{f}$ converges from both the left and the right sides. These conditions yield a linear system of equations

$$2a_{2,l}t_{l+1} + a_{1,l} = \dot{f}\left(t_{l+1}\right), \tag{421}$$

and

$$2a_{2,l}t_l + a_{1,l} = \dot{f}\left(t_l\right) \tag{422}$$

whose solutions are the coefficients $a_{1,l}$ and $a_{2,l}$,

$$a_{2,l} = \frac{\dot{f}\left(t_{l+1}\right) - \dot{f}\left(t_l\right)}{2\Delta t_l}, \tag{423}$$

and

$$a_{1,l} = \dot{f}\left(t_{l+1}\right) - 2a_{2,l}t_{l+1}. \tag{424}$$

Using the mean value theorem for integrals, one gets the coefficient $a_{0,l}$ to make the integral of the representation equal to the integral of the represented function at each point t_k. The integral of the representation is

$$\int_{t_l}^{t_{l+1}} a_{2,l}t^2 + a_{1,l}t + a_{0,l}\,\mathrm{d}t = \frac{a_{2,l}}{3}\left(t_{l+1}^3 - t_l^3\right) + \frac{a_{1,l}}{2}\left(t_{l+1}^2 - t_l^2\right) + a_{0,l}\left(t_{l+1} - t_l\right), \tag{425}$$

which, comparing to the integral of the represented function, reduces to

$$a_{0,l} = \frac{1}{\Delta t_l} \int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \frac{1}{\Delta t_l} \frac{a_{2,l}}{3} \left(t_{l+1}^3 - t_l^3\right) - \frac{1}{\Delta t_l} \frac{a_{1,l}}{2} \left(t_{l+1}^2 - t_l^2\right). \tag{426}$$

Turning these equations in terms of $\hat{c}_{k,j}$ yields

$$\hat{c}_{l,2} = \frac{\dot{f}(t_{l+1}) - \dot{f}(t_l)}{2\Delta t_l} - \sum_{k=0}^{l-1} \hat{c}_{k,2}, \tag{427}$$

$$\hat{c}_{l,1} = \dot{f}(t_{l+1}) - 2a_{2,l} t_{l+1} - \sum_{k=0}^{l-1} \hat{c}_{k,1} \tag{428}$$

and

$$\hat{c}_{l,0} = \frac{1}{\Delta t_l} \int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \frac{1}{\Delta t_l} \frac{a_{2,l}}{3} \left(t_{l+1}^3 - t_l^3\right) - \frac{1}{\Delta t_l} \frac{a_{1,l}}{2} \left(t_{l+1}^2 - t_l^2\right) - \sum_{k=0}^{l-1} \hat{c}_{k,0}. \tag{429}$$

4.1.2 n-dimensional functions $\mathbf{f}: V \to V^n$

In an analogous manner to one-dimensional functions, let approximate n-dimensional functions $\mathbf{f}: V \to V^n$ extending Eq. (408). For this purpose, let a vector function be written as linear combination of the vectors of the canonical base, $\mathbf{e}_j$,

$$\mathbf{f}(t) = g_1(t)\mathbf{e}_1 + g_2(t)\mathbf{e}_2 + \ldots + g_n(t)\mathbf{e}_n, \tag{430}$$

where each functional coefficient $g_j(t)$, $j \in \{1, 2, \ldots, n\}$, can be represented using HS,

$$g_j(t) \approx \hat{g}_j(t) = \sum_{k=0}^{n_k} \sum_{m=0}^{n_{km}} \hat{c}_{j,k,m} t^m \mathscr{H}\left(t - t_{j,k}\right). \tag{431}$$

Hence, the HS is extended to the n-dimensional codomain, V^n, by simply representing each component of the vector function $\mathbf{f}$ independently. Analogously to one-dimensional representation, there will be different orders of approximation as well. For the purpose of illustrating the process for the n-dimensional case, first order only will be addressed, since it both shows how the procedure is carried out and it is the most prominent approximation, as following results will demonstrate.

4.1.2.1 First order approximation

Assuming that functions $g_j(t)$ are written in approximate form as in Eq. (431) and particularizing for $n_{km} = 1$, Equation (431) becomes

$$g_j(t) \approx \hat{g}_j(t) = \sum_{k=0}^{n_k} \left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t\right) \mathscr{H}\left(t - t_{j,k}\right). \tag{432}$$

Due to the Heaviside steps, the approximation at interval $t \in [t_l, t_{l+1}]$ is given by

$$\hat{g}_j(t) = \sum_{k=0}^{l} \hat{c}_{j,k,0} + \sum_{k=0}^{l} \hat{c}_{j,k,1}t = a_{0jl} + a_{1,j,l}t, \;\; t_l \leq t \leq t_{l+1}. \tag{433}$$

The slope $a_{1,j,l}$ of this polynomial can be tailored to be equal to

$$a_{1,j,l} = \frac{g_j\left(t_{l+1}\right) - g_j\left(t_l\right)}{\Delta t_l}, \tag{434}$$

whereas coefficient $a_{0,j,l}$ is obtained by the equality between the integrals of the original function and its representation through HS,

$$\int_{t_l}^{t_{l+1}} \hat{g}_j(t)\,\mathrm{d}t = a_{0,j,l}\Delta t_l + \frac{a_{1,j,l}}{2}\left(t_{l+1}^2 - t_l^2\right) = \int_{t_l}^{t_{l+1}} g_j(t)\,\mathrm{d}t, \tag{435}$$

such that,

$$a_{0,j,l} = \frac{1}{\Delta t_l}\int_{t_l}^{t_{l+1}} g_j(t)\,\mathrm{d}t - \frac{a_{1,j,l}}{2\Delta t_l}\left(t_{l+1}^2 - t_l^2\right). \tag{436}$$

Thus, by using Eq. (433), the coefficients are evaluated as

$$\hat{c}_{j,l,0} = a_{0,j,l} - \sum_{k=0}^{l-1} \hat{c}_{j,k,0} \tag{437}$$

and

$$\hat{c}_{j,l,1} = a_{1,j,l} - \sum_{k=0}^{l-1} \hat{c}_{j,k,1}. \tag{438}$$

4.2 USING THE HEAVISIDE SERIES TO SOLVE ODES

Systems of coupled second order ODEs with constant matrix coefficients are represented by Eq. (1). As matrix coefficient $\mathbf{M}$ is non-singular, it its possible to rewrite this Equation as

$$\mathbf{I}\ddot{\mathbf{y}}(t) + \bar{\mathbf{C}}\dot{\mathbf{y}}(t) + \bar{\mathbf{K}}\mathbf{y}(t) = \bar{\mathbf{f}}(t) \tag{439}$$

where $\bar{\mathbf{C}} = \mathbf{M}^{-1}\mathbf{C}$, $\bar{\mathbf{K}} = \mathbf{M}^{-1}\mathbf{K}$ and $\bar{\mathbf{f}}(t) = \mathbf{M}^{-1}\mathbf{f}$(t). The Generalized Integrating Factor, an analytical approach to solve this problem, was proposed in Chapter 3. The general solution for ODEs with constant coefficients is given by

$$\mathbf{y}(t) = \underbrace{\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \exp\left(-\mathbf{F}_{2,1,1} t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}}\, \mathrm{d}t\, \mathrm{d}t}_{\mathbf{y_p}(t)} +$$
$$\underbrace{\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \exp\left(-\mathbf{F}_{2,1,1} t\right) \mathbf{C}_2\, \mathrm{d}t + \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \mathbf{C}_1}_{\mathbf{y_h}(t)}, \tag{440}$$

where $\mathbf{y_p}(t)$ is the particular solution due to the excitation vector $\mathbf{f}(t)$, $\mathbf{y_h}(t)$ is the homogeneous solution and $\mathbf{C}_1$ and $\mathbf{C}_2$ are vectors of integration constants.

When matrices $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute, as is the case with proportional damping and other damping models that yield classic normal modes (ADHIKARI, 2006), the two terms in Eq. (513) are simplified to

$$\mathbf{y_h}(t) = \exp\left(-\mathbf{F}_{2,1,1} t\right) \mathbf{C}_2 + \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \mathbf{C}_1, \tag{441}$$

and

$$\mathbf{y_p}(t) = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}}(t)\, \mathrm{d}t\, \mathrm{d}t. \tag{442}$$

Following Chapter 3, matrix $\mathbf{F}_{2,1,1}$ is the result of

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{0}, \tag{443}$$

whose solution is

$$\mathbf{F}_{2,1,1} = \frac{1}{2}\bar{\mathbf{C}} + \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}. \tag{444}$$

Thus, analytical solutions can be found when the convolution in Eq. (442) can be evaluated in closed form, as it is shown in Chapter 3 for different families of excitation functions. Those analytical solutions are continuous in time (not a discrete solution) and do not depend on the level of damping if the matrix $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is non singular (conditions for the singularity of this matrix are given in Appendix C.1). Other important aspect when comparing to the numerical procedures it the fact that the complete solution is split in its homogeneous and particular parts. The homogeneous solution does not depend on the excitation $\bar{\mathbf{f}}(t)$ and is always analytic.

For the particular solution, nonetheless, sometimes one is interested in using excitations $\bar{\mathbf{f}}(t)$:

1. only known at discrete time points $t_{j,k}$;

2. hard to evaluate by convolution, although the integral itself can be easy to evaluate (at least numerically).

Both cases can be addressed using HS, since the convolution for Heaviside steps multiplied by polynomials has analytical solution, as observed in Chapter 2 and in Chapter 3.

Thereby, the use of the Generalized Integrating Factor approach is straightforward even beyond the subspace of excitation functions whose convolution is simply calculated. Nevertheless, it can be systematically extended for the two problematic cases referred previously, as summarized in Fig. 16.

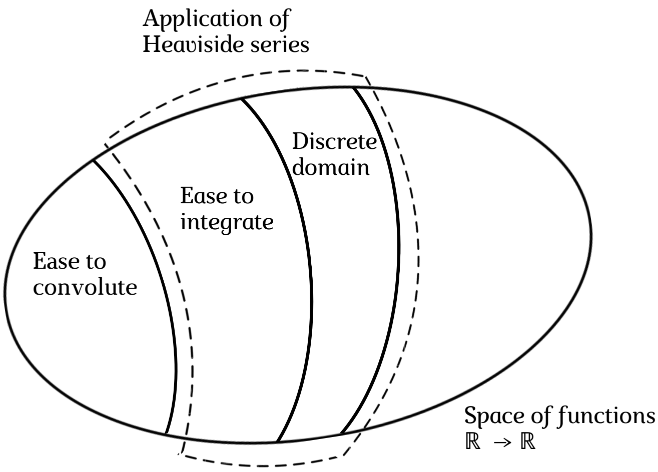

Figure 16 – Diagram of the extension of the function space covered by the use of HS representation as excitation function to the Generalized Integrating Factor. Verônica Herbst Pazda, undergraduate student of the department, is acknowledged for making this image by request of the author.

The use of the HS as excitation to evaluate the particular response using the Generalized Integrating Factor is discussed in the following.

4.2.1 Particular solution due to Heaviside Series in n-DOF problems

Let the normalized excitation vector be defined using Eq. (430),

$$\bar{\mathbf{f}}(t) = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1 + \ldots + g_n(t)\mathbf{v}_n, \tag{445}$$

with

$$g_j(t) = \sum_{k=0}^{n_k} \mathscr{H}\left(t - t_{j,k}\right) f_{j,k}(t), \tag{446}$$

where $t_{j,k}$ are n_k discrete time points for each DOF j and $f_{j,k}$ are functional coefficients.

The particular solution for this class of excitation was shown to be, Chapter 3,

$$\mathbf{y_p}(t) = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \mathscr{H}\left(t - t_{j,k}\right) \int_{t_{j,k}}^{t} \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int_{t_{j,k}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) f_{jk}(t)\, \mathrm{d}t\, \mathrm{d}t\, \mathbf{v}_j, \tag{447}$$

depending on the function that multiplies the Heaviside function. The closed-form particular solution when $f_{j,k}(t)$ are polynomials was obtained in Chapter 3, such that one can combine this solution with the approximation provided by the Heaviside series to devise a general procedure to extend the applicability of the Generalized Integrating Factor approach.

If $g_j(t)$, Eq. (446), is represented by a second order Heaviside Series

$$g_j(t) = \sum_{k=0}^{n_k} \left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1} t + \hat{c}_{jk2} t^2\right) \mathscr{H}\left(t - t_{j,k}\right), \tag{448}$$

and if $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is non-singular (C.1), the particular solution to Eq. (447), according to Chapter 3, is given by

$$\begin{aligned}
\mathbf{y_p}(t) = \sum_{j=1}^{n} \sum_{k=0}^{n_k} \mathscr{H}\left(t - t_{j,k}\right) \Big\{ & \hat{c}_{jk2} t^2 \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \mathbf{F}_{2,1,1}^{-1} + t \Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \\
& \left(\hat{c}_{j,k,1} \mathbf{F}_{2,1,1}^{-1} - 2\hat{c}_{jk2} \mathbf{F}_{2,1,1}^{-2}\right) - 2\hat{c}_{jk2} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2} \mathbf{F}_{2,1,1}^{-1} \Big) + 2\hat{c}_{jk2} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-3} \mathbf{F}_{2,1,1}^{-1} \\
& - \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2} \left(\hat{c}_{j,k,1} \mathbf{F}_{2,1,1}^{-1} - 2\hat{c}_{jk2} \mathbf{F}_{2,1,1}^{-2}\right) + \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \Big(2\hat{c}_{jk2} \mathbf{F}_{2,1,1}^{-3} \\
& - \hat{c}_{j,k,1} \mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,0} \mathbf{F}_{2,1,1}^{-1} \Big) + \exp\left(\mathbf{F}_{2,1,1} \left(t_{j,k} - t\right)\right) \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \\
& \Big(- \left(\hat{c}_{jk2} t_{j,k}^2 + \hat{c}_{j,k,1} t_{j,k} + \hat{c}_{j,k,0}\right) \mathbf{F}_{2,1,1}^{-1} + \left(2\hat{c}_{jk2} t_{j,k} + \hat{c}_{j,k,1}\right) \mathbf{F}_{2,1,1}^{-2} - 2\hat{c}_{jk2} \mathbf{F}_{2,1,1}^{-3} \Big) \\
& + \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] \left(t_{j,k} - t\right)\right) \Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \Big(\mathbf{F}_{2,1,1}^{-1} \left(-\hat{c}_{jk2} t_{j,k}^2 - \hat{c}_{j,k,1} t_{j,k} - \hat{c}_{j,k,0}\right) \\
& + \mathbf{F}_{2,1,1}^{-2} \left(2\hat{c}_{jk2} t_{j,k} + \hat{c}_{j,k,1}\right) - 2\hat{c}_{jk2} \mathbf{F}_{2,1,1}^{-3} \Big) + \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2} \Big(\mathbf{F}_{2,1,1}^{-1} \left(2\hat{c}_{jk2} t_{j,k} + \hat{c}_{j,k,1}\right) \\
& - 2\hat{c}_{jk2} \mathbf{F}_{2,1,1}^{-2} \Big) - 2\hat{c}_{jk2} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-3} \mathbf{F}_{2,1,1}^{-1} - \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \Big(-\mathbf{F}_{2,1,1}^{-1} \Big(\hat{c}_{jk2} t_{j,k}^2 \\
& + \hat{c}_{j,k,1} t_{j,k} + \hat{c}_{j,k,0} \Big) + \mathbf{F}_{2,1,1}^{-2} \left(2\hat{c}_{jk2} t_{j,k} + \hat{c}_{j,k,1}\right) - 2\hat{c}_{jk2} \mathbf{F}_{2,1,1}^{-3} \Big) \Big) \Big\} \mathbf{M}^{-1} \mathbf{e}_j.
\end{aligned} \tag{449}$$

otherwise, if $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is indeed singular (Appendix C.1), the exponential map over this matrix should be evaluated using Jordan canonical form to assess its integral (HIGHAM, 2008).

As discussed in Chapter 3, the cost of evaluation of the particular response given by Eq. (449) increases dramatically with the order of the polynomial, as shown in Tab. 2. Thus, a good compromise relationship is found using a first order approximation, resulting in

$$\begin{aligned}\mathbf{y_p}(t) = \sum_{j=1}^{n}\sum_{k=0}^{n_k} \mathscr{H}\left(t-t_{j,k}\right)\Big\{\hat{c}_{j,k,1}\Big(t\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\Big)\mathbf{F}_{2,1,1}^{-1} \\ +\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big(-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}\Big) \\ +\exp\left(\mathbf{F}_{2,1,1}\left(t_{j,k}-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\Big(-\left(\hat{c}_{j,k,1}t_{j,k}+\hat{c}_{j,k,0}\right)\mathbf{F}_{2,1,1}^{-1}+\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}\Big) \\ +\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{j,k}-t\right)\right)\Big\{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big(\mathbf{F}_{2,1,1}^{-1}\left(-\hat{c}_{j,k,1}t_{j,k}-\hat{c}_{j,k,0}\right)+\mathbf{F}_{2,1,1}^{-2}\hat{c}_{j,k,1}\Big) \\ +\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\Big(\mathbf{F}_{2,1,1}^{-1}\hat{c}_{j,k,1}\Big)-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\Big(-\mathbf{F}_{2,1,1}^{-1}\left(\hat{c}_{j,k,1}t_{j,k}+\hat{c}_{j,k,0}\right)+ \\ \mathbf{F}_{2,1,1}^{-2}\hat{c}_{j,k,1}\Big)\Big\}\Big\}\mathbf{M}^{-1}\mathbf{e}_j.\end{aligned} \tag{450}$$

For summarizing, the particular solution due to Heaviside excitation with first order polynomials can be obtained from Eq. (450), where coefficients $\hat{c}_{j,k,0}$ and $\hat{c}_{j,k,1}$ are evaluated according to Eqs. (437) and (438) by pre-processing the loading data in advance. The complete solution is comprised of the sum of this particular response and the analytical solution of the homogeneous response, Eq. (441).

Table 2 – Matrix inverses for each approximation order.

Zero order	$\mathbf{F}_{2,1,1}^{-1}$	$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}$	$\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}$
First order	$\mathbf{F}_{2,1,1}^{-1}$	$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}$	$\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}$
	$\mathbf{F}_{2,1,1}^{-2}$	$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}$	
Second order	$\mathbf{F}_{2,1,1}^{-1}$	$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}$	$\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}$
	$\mathbf{F}_{2,1,1}^{-2}$	$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}$	
	$\mathbf{F}_{2,1,1}^{-3}$	$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-3}$	

4.2.2 Properties of the solution using HS

One can devise some interesting aspects of the solution provided by Eq. (450).

4.2.2.1 Accuracy

Particular solution provided by Eq. (450) is analytical due to approximated non-homogeneous terms, as the only approximations are introduced in the computation of coefficients $\hat{c}_{j,k,0}$ and $\hat{c}_{j,k,1}$, associated to the approximation of the excitations $g_j(t)$. Also, there is no assumption on the behavior of the solution between discrete time points, like in implicit and explicit numerical methods.

The homogeneous solution, Eq. (441), does not depend on such approximations. This is another advantage when comparing the proposed approach with purely numerical methods, since the complete solution is a linear combination of these two terms and one of these terms is always analytical.

4.2.2.2 Integration

The integrals of $g_j(t)$ from t_l to t_{l+1} can be carried out analytically if $g_j(t)$ is easy to integrate. This can be used to avoid numerical errors due to numerical integration. If the integral cannot be evaluated analytically, one can resort to numerical approximations.

One interesting result is the fact that if $g_j(t)$ is known only at discrete times t_l and t_{l+1}, then, the integrals can be approximated by using a trapezoidal rule

$$\int_{t_l}^{t_{l+1}} g_j(t)\,\mathrm{d}t \approx (t_{l+1}-t_l)\left(g_j(t_l)+\frac{1}{2}\left(\left(g_j(t_{l+1})-g_j(t_l)\right)\right)\right), \tag{451}$$

extending the approach to discrete excitations.

4.2.2.3 Preservation of Impulse

The integrals of the represented functions, $g_j(t)$, and of the HS approximations are equal at the singularity points $t_{j,k}$. Therefore, the impulse I and variation of the linear momentum Δp for each DOF j are conserved, (SERWAY; JEWETT, 2004),

$$\int_{t_i}^{t_j} g_j(t)\,\mathrm{d}t = I = \Delta p, \tag{452}$$

regardless of the time discretization.

4.2.2.4 Integral measures

Still regarding integral properties, integrating Eq. (1) between the singularity points, t_i and t_j, of the HS yields

$$\mathbf{M}\left(\dot{\mathbf{y}}\left(t_j\right)-\dot{\mathbf{y}}\left(t_i\right)\right)+\mathbf{C}\left(\mathbf{y}\left(t_j\right)-\mathbf{y}\left(t_i\right)\right)+\mathbf{K}\int_{t_i}^{t_j}\mathbf{y}\,\mathrm{d}t=\int_{t_i}^{t_j}\mathbf{f}\,\mathrm{d}t. \tag{453}$$

Thus, the integral of the complete response $\mathbf{y}(t)$ is given by

$$\int_{t_i}^{t_j}\mathbf{y}\,\mathrm{d}t=\mathbf{K}^{-1}\left(\underbrace{\int_{t_i}^{t_j}\mathbf{f}\,\mathrm{d}t}_{\Gamma_1}+\underbrace{\mathbf{M}\left(\dot{\mathbf{y}}\left(t_i\right)-\dot{\mathbf{y}}\left(t_j\right)\right)+\mathbf{C}\left(\mathbf{y}\left(t_i\right)-\mathbf{y}\left(t_j\right)\right)}_{\Gamma_2}\right). \tag{454}$$

One can observe that the accuracy of the integral in the left hand side of Eq. (454) depends upon the integral of the excitation function itself, Γ_1, and upon the accuracy of the response and its first time derivative Γ_2. Therefore, for applications where the integral of the response matters, HS present another advantage, since Γ_1, the impulse of the force, is satisfied by the definition of the series coefficients.

4.2.2.5 Computational cost

As one can observe in Eq. (449) and in Tab. 2, the matrix cost of solutions due to Heaviside steps multiplied by polynomials does not depend on the number of excitation terms nor on the number of excited degrees of freedom. Hence, HS present an option to reduce matrix computation cost compared to the analytical solutions provided in Chapter 3, since its main cost is tied to the approximation order only. This is an important contrast with particular solutions due to periodic excitation functions discussed in Chapter 3, for instance, since the matrix operations depend on the number of frequencies present in the signal.

The following section is devoted to present an efficient numerical implementation of the proposed procedure, when compared to the direct evaluation of Eq. (450) and Eq. (441), without loss of accuracy.

4.3 EFFICIENT EVALUATION OF THE ODES RESPONSE USING FIRST ORDER HS

In order to evaluate the solution of system of coupled ODEs using first order HS in an efficient fashion, let Equation (450) be separated in three parts and let all matrix operations be expanded as

$$\begin{aligned}
\mathbf{y_p}(t) = \sum_{j=1}^{n}\sum_{k=0}^{n_k}\mathscr{H}\left(t-t_{j,k}\right)\Bigg[&\hat{c}_{j,k,1}t\underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_1} \\
&-\hat{c}_{j,k,1}\underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_2} \\
&-\hat{c}_{j,k,1}\underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_3}+\hat{c}_{j,k,0}\underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_1}\Bigg]\mathbf{e}_j \\
+\sum_{j=1}^{n}\sum_{k=0}^{n_k}\mathscr{H}\left(t-t_{j,k}\right)\exp\left(\mathbf{F}_{2,1,1}\left(t_{j,k}-t\right)\right)\Bigg[&-\hat{c}_{j,k,1}t_{j,k}\underbrace{\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_4} \\
&-\hat{c}_{j,k,0}\underbrace{\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_4}+\hat{c}_{j,k,1}\underbrace{\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_5}\Bigg]\mathbf{e}_j \\
+\sum_{j=1}^{n}\sum_{k=0}^{n_k}\mathscr{H}\left(t-t_{j,k}\right)\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{j,k}-t\right)\right)\Bigg[&-\hat{c}_{j,k,1}t_{j,k}\underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_1} \\
&-\hat{c}_{j,k,0}\underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_1}+\hat{c}_{j,k,1}\underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_3} \\
&+\hat{c}_{j,k,1}\underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_2}+\hat{c}_{j,k,1}t_{j,k}\underbrace{\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_4} \\
&+\hat{c}_{j,k,0}\underbrace{\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_4}-\hat{c}_{j,k,1}\underbrace{\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}}_{\mathbf{\Gamma}_5}\Bigg]\mathbf{e}_j.
\end{aligned} \tag{455}$$

One notices that there are, in total, 5 matrix-vector products for each j, whose matrices are $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$, $\mathbf{\Gamma}_3$, $\mathbf{\Gamma}_4$ and $\mathbf{\Gamma}_5$ and vectors are $\mathbf{e}_j$. Using inverse properties, each of these matrices can be simplified, as it will be shown in the following equations for each $\mathbf{\Gamma}_i$.

Matrix $\mathbf{\Gamma}_1$, that appears 4 times in Eq. (455), can be simplified to

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}=\left(\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right)^{-1}\mathbf{M}^{-1}=\left(\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right)^{-1}, \tag{456}$$

which, using Eq. (443), is further simplified to

$$\mathbf{\Gamma}_1=\left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1}=\mathbf{K}^{-1}=\mathbf{\Omega}_1^{-1}. \tag{457}$$

Matrix $\mathbf{\Gamma}_2$, that appears 2 times in Eq. (455), can be simplified to

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}=\left(\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{2}\right)^{-1}\mathbf{M}^{-1}=\left(\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{2}\right)^{-1}, \tag{458}$$

which, using Eq. (443), is further simplified to

$$\mathbf{\Gamma}_2=\left(\mathbf{M}\bar{\mathbf{K}}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right)^{-1}=\left(\mathbf{K}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right)^{-1}=\mathbf{\Omega}_2^{-1}. \tag{459}$$

Matrix $\mathbf{\Gamma}_3$, that appears 2 times in Eq. (455), can be simplified as

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}=\left(\mathbf{F}_{2,1,1}^{2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right)^{-1}\mathbf{M}^{-1}=\left(\mathbf{M}\mathbf{F}_{2,1,1}^{2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right)^{-1}, \tag{460}$$

which, using Eq. (443) and the commutativity of $\mathbf{F}_{2,1,1}$ with $\bar{\mathbf{K}}$, as proven in Chapter 3, is further simplified to

$$\mathbf{\Gamma}_3=\left(\mathbf{M}\mathbf{F}_{2,1,1}\bar{\mathbf{K}}\right)^{-1}=\left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1}=\left(\mathbf{K}\mathbf{F}_{2,1,1}\right)^{-1}=\mathbf{\Omega}_3^{-1}. \tag{461}$$

Matrix $\mathbf{\Gamma}_4$, that appears 4 times in Eq. (455), can be simplified as

$$\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}=\left(\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right)^{-1}\mathbf{M}^{-1}=\left(\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right)^{-1}, \tag{462}$$

which, using Eq. (444), is further simplified to

$$\mathbf{\Gamma}_4=\left(\mathbf{M}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right)^{-1}=\left(\mathbf{M}\left[\bar{\mathbf{K}}-\mathbf{F}_{2,1,1}^{2}\right]\right)^{-1}=\left(\mathbf{K}-\mathbf{M}\mathbf{F}_{2,1,1}^{2}\right)^{-1}=\mathbf{\Omega}_4^{-1}. \tag{463}$$

Matrix $\mathbf{\Gamma}_5$, that appears 2 times in Eq. (455), can be simplified as

$$\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}=\left(\mathbf{F}_{2,1,1}^{2}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right)^{-1}\mathbf{M}^{-1}=\left(\mathbf{M}\mathbf{F}_{2,1,1}^{2}\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\right)^{-1}, \tag{464}$$

which, using Eq. (444) and the commutativity of $\mathbf{F}_{2,1,1}$ with $\bar{\mathbf{K}}$, as proven in Chapter 3, is further simplified to

$$\mathbf{\Gamma}_5 = \left(\mathbf{MF}_{2,1,1}^2\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]\right)^{-1} = \left(\mathbf{MF}_{2,1,1}\bar{\mathbf{K}} - \mathbf{MF}_{2,1,1}^3\right)^{-1} = \left(\mathbf{KF}_{2,1,1} - \mathbf{MF}_{2,1,1}^3\right)^{-1}$$
$$= \left(\mathbf{\Omega}_4\mathbf{F}_{2,1,1}\right)^{-1} = \left(\mathbf{\Omega}_3 - \mathbf{MF}_{2,1,1}^3\right)^{-1} = \mathbf{\Omega}_5^{-1}. \quad (465)$$

Henceforth, all inverse operations can be carried out using a linear system of equations,

$$\mathbf{x}_{i,j} = \mathbf{\Omega}_i \backslash \mathbf{e}_j, \quad i = 1,2,3,4,5. \quad (466)$$

where $\backslash$ indicates the solution of a linear system $\mathbf{\Omega}_i\mathbf{x}_{i,j} = \mathbf{e}_j$, and $\mathbf{x}_{i,j}$ is the solution. Thus, substituting those results into Eq. (455), it yields

$$\mathbf{y_p}(t) = \sum_{j=1}^{n}\left[\sum_{k=0}^{n_k}\mathscr{H}\left(t - t_{j,k}\right)\left[\left(\hat{c}_{j,k,1}t + \hat{c}_{j,k,0}\right)\mathbf{x}_{1,j} - \hat{c}_{j,k,1}\mathbf{x}_{2,j} - \hat{c}_{j,k,1}\mathbf{x}_{3,j}\right]\right.$$
$$+\sum_{k=0}^{n_k}\mathscr{H}\left(t - t_{j,k}\right)\exp\left(\mathbf{F}_{2,1,1}\left(t_{j,k} - t\right)\right)\left[-\left(\hat{c}_{j,k,1}t_{j,k} + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j} + \hat{c}_{j,k,1}\mathbf{x}_{5,j}\right]$$
$$+\sum_{k=0}^{n_k}\mathscr{H}\left(t - t_{j,k}\right)\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\left(t_{j,k} - t\right)\right)\left[-\left(\hat{c}_{j,k,1}t_{j,k} + \hat{c}_{j,k,0}\right)\mathbf{x}_{1j}\right.$$
$$\left.\left.+\hat{c}_{j,k,1}\mathbf{x}_{2,j} + \hat{c}_{j,k,1}\mathbf{x}_{3,j} + \left(\hat{c}_{j,k,1}t_{j,k} + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j} - \hat{c}_{j,k,1}\mathbf{x}_{5,j}\right]\right]. \quad (467)$$

To this point, the matrix operations with the matrix coefficients that multiply the polynomial and exponential terms were simplified, with great reduction in expected computation cost. Let further considerations regarding the exponential maps be made. For this purpose, let the time points, $t_{j,k}$, be equally spaced by a time step Δt and the initial time, t_0, coincide with $t_{j,0}$ (the same discrete times are used for all excited DOFs j). Finally, let the response, $\mathbf{y_p}$, be evaluated in the same discrete set of time points, *i.e.* $t_k = t_{j,k}$. As the same time discretization is made for all of the excited DOFs, the summations in j and in k can be swapped,

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n}\mathscr{H}\left(t - t_k\right)\left[t\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j}\right)\right]$$
$$+\sum_{k=0}^{n_k}\sum_{j=1}^{n}\mathscr{H}\left(t - t_k\right)\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t\right)\right)\left[-\left(\hat{c}_{j,k,1}t_k + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j} + \hat{c}_{j,k,1}\mathbf{x}_{5,j}\right]$$
$$+\sum_{k=0}^{n_k}\sum_{j=1}^{n}\mathscr{H}\left(t - t_k\right)\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\left(t_k - t\right)\right)\left[\left(\hat{c}_{j,k,1}t_k + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}\right.$$
$$\left.+\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j} - \mathbf{x}_{5,j}\right) - t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j} - \hat{c}_{j,k,0}\mathbf{x}_{1,j}\right]. \quad (468)$$

Let Equation (468) be evaluated at time t_i, where t_i belongs to the set of discrete time points t_k. Using the definition of the Heaviside step function, Equation (468) is simplified and some structures are highlighted,

$$\mathbf{y_p}(t_i) = \sum_{k=0}^{i}\sum_{j=1}^{n}\left[t_i\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j}\right)\right]$$
$$\underbrace{+\sum_{k=0}^{i}\sum_{j=1}^{n}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_i\right)\right)\left[-\left(\hat{c}_{j,k,1}t_k + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j} + \hat{c}_{j,k,1}\mathbf{x}_{5,j}\right]}_{\boldsymbol{\gamma}_{1,i}}$$
$$+\underbrace{\left\{\begin{array}{c}\sum_{k=0}^{i}\sum_{j=1}^{n}\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_i)\right)\left[\left(\hat{c}_{j,k,1}t_k + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}\right. \\ \left. +\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j} - \mathbf{x}_{5,j}\right) - t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j} - \hat{c}_{j,k,0}\mathbf{x}_{1,j}\right]\end{array}\right\}}_{\boldsymbol{\gamma}_{2,i}}.$$

Now, to assess how the method progress from one point to the next, let Equation (468) be evaluated at time t_{i+1}, it follows from the definition of the Heaviside step function that

$$\begin{aligned}\mathbf{y_p}(t_{i+1}) &= \sum_{k=0}^{i+1}\sum_{j=1}^{n}\left[t_{i+1}\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j}\right)\right] \\ &+\sum_{k=0}^{i+1}\sum_{j=1}^{n}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_{i+1}\right)\right)\left[-\left(\hat{c}_{j,k,1}t_k + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j} + \hat{c}_{j,k,1}\mathbf{x}_{5,j}\right] \\ &+\sum_{k=0}^{i+1}\sum_{j=1}^{n}\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_{i+1})\right)\left[\left(\hat{c}_{j,k,1}t_k + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}\right. \\ &\left. +\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j} - \mathbf{x}_{5,j}\right) - t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j} - \hat{c}_{j,k,0}\mathbf{x}_{1,j}\right],\end{aligned} \tag{469}$$

taking out the $(i+1)$-th term of each summation in k, it yields

$$\begin{aligned}\mathbf{y_p}(t_{i+1}) &= \sum_{j=1}^{n}\left[t_{i+1}\hat{c}_{j,i+1,1}\mathbf{x}_{1,j} + \hat{c}_{j,i+1,0}\mathbf{x}_{1,j} - \hat{c}_{j,i+1,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j}\right)\right. \\ &+\left[-\left(\hat{c}_{j,i+1,1}t_{i+1} + \hat{c}_{j,i+1,0}\right)\mathbf{x}_{4,j} + \hat{c}_{j,i+1,1}\mathbf{x}_{5,j}\right] + \left[\left(\hat{c}_{j,i+1,1}t_{i+1} + \hat{c}_{j,i+1,0}\right)\mathbf{x}_{4,j}\right. \\ &\left.\left. +\hat{c}_{j,i+1,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j} - \mathbf{x}_{5,j}\right) - t_{i+1}\hat{c}_{j,i+1,1}\mathbf{x}_{1,j} - \hat{c}_{j,i+1,0}\mathbf{x}_{1,j}\right]\right] \\ &+\sum_{k=0}^{i}\sum_{j=1}^{n}\left[t_{i+1}\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j}\right)\right] \\ &+\sum_{k=0}^{i}\sum_{j=1}^{n}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_{i+1}\right)\right)\left[-\left(\hat{c}_{j,k,1}t_k + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j} + \hat{c}_{j,k,1}\mathbf{x}_{5,j}\right] \\ &+\sum_{k=0}^{i}\sum_{j=1}^{n}\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_{i+1})\right)\left[\left(\hat{c}_{j,k,1}t_k + \hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}\right. \\ &\left. +\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j} - \mathbf{x}_{5,j}\right) - t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j} - \hat{c}_{j,k,0}\mathbf{x}_{1,j}\right].\end{aligned} \tag{470}$$

All the terms out of the summations in k in Eq. (470) cancel themselves out. Thus, since $t_{i+1} = t_i + \Delta t$, it is possible to rewrite Eq. (470) as

$$\mathbf{y_p}(t_{i+1}) = t_{i+1}\underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}}_{\mathbf{w}_{1,i}} + \underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}}_{\mathbf{w}_{2,i}} - \underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}\right)}_{\mathbf{w}_{3,i}}$$
$$+\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\underbrace{\left\{\sum_{k=0}^{i}\sum_{j=1}^{n}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_i\right)\right)\left[-\left(\hat{c}_{j,k,1}t_k+\hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\mathbf{x}_{5,j}\right]\right\}}_{\boldsymbol{\gamma}_{1,i}}$$
$$+\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\underbrace{\left\{\begin{matrix}\sum_{k=0}^{i}\sum_{j=1}^{n}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t_i\right)\right)\left[\left(\hat{c}_{j,k,1}t_k+\hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}\right.\\ \left.+\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}-\mathbf{x}_{5,j}\right)-t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}-\hat{c}_{j,k,0}\mathbf{x}_{1,j}\right]\end{matrix}\right\}}_{\boldsymbol{\gamma}_{2,i}}. \quad (471)$$

Finally, the response at time t_{i+1} can be rewritten in a simpler form

$$\mathbf{y_p}(t_{i+1}) = t_{i+1}\mathbf{w}_{1,i}+\mathbf{w}_{2,i}-\mathbf{w}_{3,i}+\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}+\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i}. \quad (472)$$

Vectors $\boldsymbol{\gamma}_{1,i}$ and $\boldsymbol{\gamma}_{2,i}$ were defined to establish a link between $\mathbf{y_p}(t_{i+1})$ and $\mathbf{y_p}(t_i)$, for summations in both vectors are from $k=0$ up to $k=i$. Therefore, $\mathbf{y_p}(t_{i+2})$ will be related to $\mathbf{y_p}(t_{i+1})$ through $\boldsymbol{\gamma}_{1,i+1}$ and $\boldsymbol{\gamma}_{2,i+1}$ and so on, recursively. Hence, an update rule for vector $\boldsymbol{\gamma}$ must be derived and this can be achieved with Eq. (468) evaluated at t_{i+2}. Again the terms with $k=i+2$ are taken out of the summations just like in Eq. (470) and the summations over exponentials can be simplified to

$$\mathbf{y_p}(t_{i+2}) = \sum_{j=1}^{n}\left[t_{i+2}\hat{c}_{j,i+2,1}\mathbf{x}_{1,j}+\hat{c}_{j,i+2,0}\mathbf{x}_{1,j}-\hat{c}_{j,i+2,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}\right)+\left[-\left(\hat{c}_{j,i+2,1}t_{i+2}\right.\right.\right.$$
$$\left.+\hat{c}_{j,i+2,0}\right)\mathbf{x}_{4,j}+\hat{c}_{j,i+2,1}\mathbf{x}_{5,j}\Big]+\Big[\left(\hat{c}_{j,i+2,1}t_{i+2}+\hat{c}_{j,i+2,0}\right)\mathbf{x}_{4,j}+\hat{c}_{j,i+2,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}-\mathbf{x}_{5,j}\right)$$
$$\left.\left.-t_{i+2}\hat{c}_{j,i+2,1}\mathbf{x}_{1,j}-\hat{c}_{j,i+2,0}\mathbf{x}_{1,j}\right]\right]+\sum_{k=0}^{i+1}\sum_{j=1}^{n}\left[t_{i+2}\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\hat{c}_{j,k,0}\mathbf{x}_{1,j}-\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}\right)\right]$$
$$+\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\underbrace{\sum_{k=0}^{i+1}\sum_{j=1}^{n}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_{i+1}\right)\right)\left[-\left(\hat{c}_{j,k,1}t_k+\hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\mathbf{x}_{5,j}\right]}_{\boldsymbol{\gamma}_{1,i+1}}+$$
$$\left.\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\underbrace{\left\{\begin{matrix}\sum_{k=0}^{i+1}\sum_{j=1}^{n}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t_{i+1}\right)\right)\left[\left(\hat{c}_{j,k,1}t_k+\hat{c}_{j,k,0}\right)\right.\\ \left.\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}-\mathbf{x}_{5,j}\right)-t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}-\hat{c}_{j,k,0}\mathbf{x}_{1,j}\right]\end{matrix}\right\}}_{\boldsymbol{\gamma}_{2,i+1}}\right]. \quad (473)$$

The terms with index $i+2$ in Eq. (473) cancel each other out. Taking the terms with $k=i+1$ out of the summations with exponential maps, taking the terms with $k=i+1$ out of the summation without exponential maps, and recollecting that $t_{i+1}=t_i+\Delta t$, Equation (473) can be rewritten as

$$\begin{aligned}
\mathbf{y_p}(t_{i+2}) &= t_{i+2}\sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{1,j}+\sum_{j=1}^{n}\hat{c}_{j,i+1,0}\mathbf{x}_{1,j}-\sum_{j=1}^{n}\hat{c}_{j,i+1,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}\right) \\
&+t_{i+2}\underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}}_{\mathbf{w}_{1,i}}+\underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}}_{\mathbf{w}_{2,i}}-\underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}\right)}_{\mathbf{w}_{3,i}}+\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right) \\
&\underbrace{\left\{\begin{array}{c}\sum_{j=1}^{n}\left[-\left(\hat{c}_{j,i+1,1}t_{i+1}+\hat{c}_{j,i+1,0}\right)\mathbf{x}_{4,j}+\hat{c}_{j,i+1,1}\mathbf{x}_{5,j}\right] \\ +\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\sum_{k=0}^{i}\sum_{j=1}^{n}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_i\right)\right)\left[-\left(\hat{c}_{j,k,1}t_k+\hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\mathbf{x}_{5,j}\right]\end{array}\right\}}_{\boldsymbol{\gamma}_{1,i+1}} \\
&+\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\underbrace{\left\{\begin{array}{c}\sum_{j=1}^{n}\left[\left(\hat{c}_{j,i+1,1}t_{i+1}+\hat{c}_{j,i+1,0}\right)\mathbf{x}_{4,j}+\hat{c}_{j,i+1,1}\left(\mathbf{x}_{2,j}\right.\right. \\ \left.\left.+\mathbf{x}_{3,j}-\mathbf{x}_{5,j}\right)-t_{i+1}\hat{c}_{j,i+1,1}\mathbf{x}_{1,j}-\hat{c}_{j,i+1,0}\mathbf{x}_{1,j}\right] \\ +\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\sum_{k=0}^{i}\sum_{j=1}^{n}\left[\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right.\right. \\ \left.\left(t_k-t_i\right)\right)\left[\left(\hat{c}_{j,k,1}t_k+\hat{c}_{j,k,0}\right)\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}-\mathbf{x}_{5,j}\right)\right. \\ \left.\left.-t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}-\hat{c}_{j,k,0}\mathbf{x}_{1,j}\right]\right]\end{array}\right\}}_{\boldsymbol{\gamma}_{2,i+1}}.
\end{aligned} \tag{474}$$

Hence, the response at time t_{i+2} can be evaluated similarly to $\mathbf{y_p}\left(t_{i+1}\right)$, Eq. (472),

$$\begin{aligned}
\mathbf{y_p}(t_{i+2}) &= t_{i+2}\underbrace{\left(\mathbf{w}_{1,i}+\underbrace{\sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{1,j}}_{\mathbf{d}_{1,i+1}}\right)}_{\mathbf{w}_{1,i+1}}+\underbrace{\left(\mathbf{w}_{2,i}+\underbrace{\sum_{j=1}^{n}\hat{c}_{j,i+1,0}\mathbf{x}_{1,j}}_{\mathbf{d}_{2,i+1}}\right)}_{\mathbf{w}_{2,i+1}} \\
&-\underbrace{\left(\mathbf{w}_{3,i}+\underbrace{\sum_{j=1}^{n}\hat{c}_{j,i+1,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}\right)}_{\mathbf{d}_{3,i+1}}\right)}_{\mathbf{w}_{3,i+1}} \\
&+\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i+1}+\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i+1}.
\end{aligned} \tag{475}$$

By comparing Eq. (472) to Eq. (475), there immediately follow update rules for vectors $\boldsymbol{\gamma}$ and $\mathbf{w}$,

$$\mathbf{w}_{1,i+1} = \mathbf{w}_{1,i} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1} \mathbf{x}_{1,j}}_{\mathbf{d}_{1,i+1}},$$

$$\mathbf{w}_{2,i+1} = \mathbf{w}_{2,i} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,0} \mathbf{x}_{1,j}}_{\mathbf{d}_{2,i+1}},$$

$$\mathbf{w}_{3,i+1} = \mathbf{w}_{3,i} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1} \left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j}\right)}_{\mathbf{d}_{3,i+1}},$$

$$\boldsymbol{\gamma}_{1,i+1} = \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i} - \underbrace{\sum_{j=1}^{n} \left(\hat{c}_{j,i+1,1} t_{i+1} + \hat{c}_{j,i+1,0}\right) \mathbf{x}_{4,j}}_{\mathbf{d}_{4,i+1}} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1} \mathbf{x}_{5,j}}_{\mathbf{d}_{5,i+1}},$$

$$\boldsymbol{\gamma}_{2,i+1} = \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i} + \underbrace{\sum_{j=1}^{n} \left(\hat{c}_{j,i+1,1} t_{i+1} + \hat{c}_{j,i+1,0}\right) \mathbf{x}_{4,j}}_{\mathbf{d}_{4,i+1}}$$

$$+ \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1} \left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j}\right)}_{\mathbf{d}_{3,i+1}} - \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1} \mathbf{x}_{5,j}}_{\mathbf{d}_{5,i+1}} - t_{i+1} \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1} \mathbf{x}_{1,j}}_{\mathbf{d}_{1,i+1}} - \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,0} \mathbf{x}_{1,j}}_{\mathbf{d}_{2,i+1}}. \tag{476}$$

Let a generic form of the summations in j be studied and expanded,

$$\sum_{j=1}^{n} b_j \mathbf{x}_{i,j} = b_1 \mathbf{x}_{i,1} + b_2 \mathbf{x}_{i,2} + \cdots + b_n \mathbf{x}_{i,n} = \begin{bmatrix} b_1 x_{i,1,1} + b_2 x_{i,2,1} + \cdots + b_n x_{i,n,1} \\ b_1 x_{i,1,2} + b_2 x_{i,2,2} + \cdots + b_n x_{i,n,2} \\ \cdots \\ b_1 x_{i,1,n} + b_2 x_{i,2,n} + \cdots + b_n x_{i,n,n} \end{bmatrix}, \tag{477}$$

where $x_{i,j,p}$ is the p-th component of the vector $\mathbf{x}_{i,j}$. Equation (477) can be expressed in matrix notation as

$$\sum_{j=1}^{n} b_j \mathbf{x}_{i,j} = \begin{bmatrix} b_1 x_{i,1,1} + b_2 x_{i,2,1} + \cdots + b_n x_{i,n,1} \\ b_1 x_{i,1,2} + b_2 x_{i,2,2} + \cdots + b_n x_{i,n,2} \\ \cdots \\ b_1 x_{i,1,n} + b_2 x_{i,2,n} + \cdots + b_n x_{i,n,n} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{x}_{i,1} & \mathbf{x}_{i,2} & \ldots & \mathbf{x}_{i,n} \end{bmatrix}}_{\mathbf{X}_i} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}}_{\mathbf{b}}. \tag{478}$$

It is important to stress that j is not a counter on the DOFs of system but a counter of **excited DOFs** of the system. For that reason, the summations in j can be substituted by the following notation,

$$\sum_{j=1}^{n} b_j \mathbf{x}_{i,j} \equiv \sum_{j\in\mathscr{S}} b_{\mathscr{S}(j)} \mathbf{x}_{i,\mathscr{S}(j)} = \underbrace{\begin{bmatrix}\mathbf{x}_{i,\mathscr{S}(1)} & \mathbf{x}_{i,\mathscr{S}(2)} & \cdots & \mathbf{x}_{i,\mathscr{S}(n_e)}\end{bmatrix}}_{\mathbf{X}_i} \underbrace{\begin{bmatrix} b_{\mathscr{S}(1)} \\ b_{\mathscr{S}(2)} \\ \cdots \\ b_{\mathscr{S}(n_e)} \end{bmatrix}}_{\mathbf{b}}, \tag{479}$$

where $\mathscr{S}$ is the set of excited DOFs, n_e is the cardinality of $\mathscr{S}$, and $\mathscr{S}(j)$ is the j-th component of $\mathscr{S}$, *i.e.* it is the j-th excited DOF that is contained in the aforementioned set. The importance of set $\mathscr{S}$ is to mathematically lay the foundation for applications where not all DOFs are excited. Therefore, the matrix $\mathbf{X}_i$ is a $n \times n_e$ matrix, whereas $\mathbf{b}$ is a $n \times 1$ vector.

Using the reasoning from Eq. (479), the vectors $\mathbf{d}$ in Eq. (476) are rewritten as

$$\begin{aligned}
\mathbf{d}_{1,i+1} &= \mathbf{X}_1 \begin{bmatrix} \hat{c}_{\mathscr{S}(1),i+1,1} \\ \hat{c}_{\mathscr{S}(2),i+1,1} \\ \cdots \\ \hat{c}_{\mathscr{S}(n_e),i+1,1} \end{bmatrix}, \\
\mathbf{d}_{2,i+1} &= \mathbf{X}_1 \begin{bmatrix} \hat{c}_{\mathscr{S}(1),i+1,0} \\ \hat{c}_{\mathscr{S}(2),i+1,0} \\ \cdots \\ \hat{c}_{\mathscr{S}(n_e),i+1,0} \end{bmatrix}, \\
\mathbf{d}_{3,i+1} &= (\mathbf{X}_2 + \mathbf{X}_3) \begin{bmatrix} \hat{c}_{\mathscr{S}(1),i+1,1} \\ \hat{c}_{\mathscr{S}(2),i+1,1} \\ \cdots \\ \hat{c}_{\mathscr{S}(n_e),i+1,1} \end{bmatrix}, \\
\mathbf{d}_{4,i+1} &= \mathbf{X}_4 \begin{bmatrix} \hat{c}_{\mathscr{S}(1),i+1,1} t_{i+1} + \hat{c}_{\mathscr{S}(1),i+1,0} \\ \hat{c}_{\mathscr{S}(2),i+1,1} t_{i+1} + \hat{c}_{\mathscr{S}(2),i+1,0} \\ \cdots \\ \hat{c}_{\mathscr{S}(n_e),i+1,1} t_{i+1} + \hat{c}_{\mathscr{S}(n_e),i+1,0} \end{bmatrix}, \\
\mathbf{d}_{5,i+1} &= \mathbf{X}_5 \begin{bmatrix} \hat{c}_{\mathscr{S}(1),i+1,1} \\ \hat{c}_{\mathscr{S}(2),i+1,1} \\ \cdots \\ \hat{c}_{\mathscr{S}(n_e),i+1,1} \end{bmatrix}.
\end{aligned} \tag{480}$$

Then, Eq. (476) is rewritten to

$$
\begin{aligned}
\mathbf{w}_{1,i+1} &= \mathbf{w}_{1,i} + \mathbf{d}_{1,i+1}, \\
\mathbf{w}_{2,i+1} &= \mathbf{w}_{2,i} + \mathbf{d}_{2,i+1}, \\
\mathbf{w}_{3,i+1} &= \mathbf{w}_{3,i} + \mathbf{d}_{3,i+1}, \\
\boldsymbol{\gamma}_{1,i+1} &= \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i} - \mathbf{d}_{4,i+1} + \mathbf{d}_{5,i+1}, \\
\boldsymbol{\gamma}_{2,i+1} &= \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i} + \mathbf{d}_{4,i+1} + \mathbf{d}_{3,i+1} - \mathbf{d}_{5,i+1} - t_{i+1}\mathbf{d}_{1,i+1} - \mathbf{d}_{2,i+1}.
\end{aligned}
\tag{481}
$$

As the response due to HS, regardless of the order used, has always homogeneous initial conditions as reasoned in Appendix C.2, there is no point in calculating the response at t_0. However, $\mathbf{w}_{1,0}$, $\mathbf{w}_{2,0}$, $\mathbf{w}_{3,0}$, $\boldsymbol{\gamma}_{1,0}$ and $\boldsymbol{\gamma}_{2,0}$ are needed for evaluating $\mathbf{y_p}(t_1)$, thus, these 4 quantities must be initialized as

$$
\begin{aligned}
\mathbf{w}_{1,0} &= \mathbf{d}_{1,0}, \\
\mathbf{w}_{2,0} &= \mathbf{d}_{2,0}, \\
\mathbf{w}_{3,0} &= \mathbf{d}_{3,0}, \\
\boldsymbol{\gamma}_{1,0} &= \mathbf{C}_2 - \mathbf{d}_{4,0} + \mathbf{d}_{5,0}, \\
\boldsymbol{\gamma}_{2,0} &= \mathbf{C}_1 + \mathbf{d}_{4,0} + \mathbf{d}_{3,0} - \mathbf{d}_{5,0} - t_0\mathbf{d}_{1,0} - \mathbf{d}_{2,0},
\end{aligned}
\tag{482}
$$

in which, the vectors $\mathbf{d}_{i,0}$ follow the definition of Eq. (480). The addition of the initial condition vectors, $\mathbf{C}_1$ and $\mathbf{C}_2$, into the $\boldsymbol{\gamma}$ vectors is due to the dependence of all of them with the exact same exponential maps. Besides, if one compares the update rules in Eq. (476) to the efficient way of computing the homogeneous response in Alg. 7, the same structure of update will be obvious. Henceforth, the homogeneous response can be added to the response due to HS excitation without any additional computational cost per iteration and, consequently, initial conditions different than the homogeneous ones can be evaluated using HS without any problem, being computationally inexpensive. This poses an advantage over the previously proposed analytical solutions using Generalized Integrating Factor, since the homogeneous solution is evaluated separately.

4.3.1 Particularization for under damped problems

Exponentials of $\mathbf{F}_{2,1,1}$ and of $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ appear in Eq. (472). As explored in Chapter 3, when underdamped problems are analyzed, both matrices are complex conjugate of one another, such that $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] = \mathbf{F}^*_{2,1,1}$ and so are their exponentials. Thus, only one of them must be calculated in such situation and Eq. (471) turns to

$$
\begin{aligned}
\mathbf{y_p}(t_{i+1}) = t_{i+1}\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j} - \sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j} + \mathbf{x}_{3,j}\right) \\
+ \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i} + \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)^*\boldsymbol{\gamma}_{2,i}.
\end{aligned}
\tag{483}
$$

From Eq. (471), it follows by inspection that if the conditions bellow hold true, then, there is complex-conjugate relationship between the vectors $\boldsymbol{\gamma}$,

$$\begin{cases} \textbf{I) } \mathbf{x}_{2,j} + \mathbf{x}_{3,j} - \mathbf{x}_{5,j} = \mathbf{x}^*_{5,j}, \\ \textbf{II) } \mathbf{x}_{1,j} - \mathbf{x}_{4,j} = \mathbf{x}^*_{4,j} \end{cases} \implies \boldsymbol{\gamma}_{2,i} = \boldsymbol{\gamma}^*_{1,i}, \tag{484}$$

Condition **I** from Eq. (484) can be expanded into

$$\mathbf{x}_{2,j} + \mathbf{x}_{3,j} - \mathbf{x}_{5,j} = \left[\boldsymbol{\Omega}_2^{-1} + \boldsymbol{\Omega}_3^{-1} - \boldsymbol{\Omega}_5^{-1}\right]\mathbf{e}_j, \tag{485}$$

which, from Eq. (458), Eq. (460) and Eq. (464), results into

$$\begin{aligned} &\left[\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1} + \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1} - \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}\right]\mathbf{e}_j \\ &\quad = \left[\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1} + \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2} - \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\right]\mathbf{M}^{-1}\mathbf{e}_j. \end{aligned} \tag{486}$$

Using $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] = \mathbf{F}^*_{2,1,1}$ and, consequently, $\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] = \left[\mathbf{F}^*_{2,1,1} - \mathbf{F}_{2,1,1}\right]$, Equation (486) can be written as

$$\begin{aligned} &\left[\left(\mathbf{F}^*_{2,1,1}\right)^{-2}\mathbf{F}_{2,1,1}^{-1} + \left(\mathbf{F}^*_{2,1,1}\right)^{-1}\mathbf{F}_{2,1,1}^{-2} - \left[\mathbf{F}^*_{2,1,1} - \mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\right]\mathbf{M}^{-1}\mathbf{e}_j \\ &= \left[\mathbf{F}_{2,1,1} - \mathbf{F}^*_{2,1,1}\right]^{-1}\left[\left[\mathbf{F}_{2,1,1} - \mathbf{F}^*_{2,1,1}\right]\left(\mathbf{F}^*_{2,1,1}\right)^{-2}\mathbf{F}_{2,1,1}^{-1} + \right. \\ &\qquad \left.\left[\mathbf{F}_{2,1,1} - \mathbf{F}^*_{2,1,1}\right]\left(\mathbf{F}^*_{2,1,1}\right)^{-1}\mathbf{F}_{2,1,1}^{-2} + \mathbf{F}_{2,1,1}^{-2}\right]\mathbf{M}^{-1}\mathbf{e}_j \\ &= \left[\mathbf{F}_{2,1,1} - \mathbf{F}^*_{2,1,1}\right]^{-1}\left[\mathbf{F}_{2,1,1}\left(\mathbf{F}^*_{2,1,1}\right)^{-2}\mathbf{F}_{2,1,1}^{-1} - \left(\mathbf{F}^*_{2,1,1}\right)^{-1}\mathbf{F}_{2,1,1}^{-1} + \mathbf{F}_{2,1,1}\left(\mathbf{F}^*_{2,1,1}\right)^{-1}\mathbf{F}_{2,1,1}^{-2}\right. \\ &\qquad \left. - \mathbf{F}_{2,1,1}^{-2} + \mathbf{F}_{2,1,1}^{-2}\right]\mathbf{M}^{-1}\mathbf{e}_j. \end{aligned} \tag{487}$$

As $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ and $\mathbf{F}_{2,1,1}$ commute, so do $\mathbf{F}_{2,1,1}$ and $\mathbf{F}^*_{2,1,1}$, hence

$$\mathbf{x}_{2,j} + \mathbf{x}_{3,j} - \mathbf{x}_{5,j} = \left[\mathbf{F}_{2,1,1} - \mathbf{F}^*_{2,1,1}\right]^{-1}\left(\mathbf{F}^*_{2,1,1}\right)^{-2}\mathbf{M}^{-1}\mathbf{e}_j. \tag{488}$$

Let the complex conjugate of $\mathbf{x}_{5,j}$,

$$\begin{aligned} \mathbf{x}^*_{5,j} &= \left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}\right)^*\mathbf{e}_j = \left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^*\right)^{-1}\left(\mathbf{F}^*_{2,1,1}\right)^{-2}\mathbf{M}^{-1}\mathbf{e}_j \\ &= \left(\left[\mathbf{F}^*_{2,1,1} - \mathbf{F}_{2,1,1}\right]^*\right)^{-1}\left(\mathbf{F}^*_{2,1,1}\right)^{-2}\mathbf{M}^{-1}\mathbf{e}_j = \left[\mathbf{F}_{2,1,1} - \mathbf{F}^*_{2,1,1}\right]^{-1}\left(\mathbf{F}^*_{2,1,1}\right)^{-2}\mathbf{M}^{-1}\mathbf{e}_j, \end{aligned} \tag{489}$$

and, thus, condition **I** is true.

Let the LHS of condition **II** be expanded,

$$\begin{aligned}\mathbf{x}_{1,j}-\mathbf{x}_{4,j}=\left[\boldsymbol{\Omega}_1^{-1}-\boldsymbol{\Omega}_4^{-1}\right]\mathbf{e}_j=\left[\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}\right.\\ \left.-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}\right]\mathbf{e}_j=\left[\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\right]\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}\mathbf{e}_j\\ =\left[\left(\mathbf{F}_{2,1,1}^{*}\right)^{-1}-\left[\mathbf{F}_{2,1,1}^{*}-\mathbf{F}_{2,1,1}\right]^{-1}\right]\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}\mathbf{e}_j\\ =\left[\mathbf{F}_{2,1,1}-\mathbf{F}_{2,1,1}^{*}\right]^{-1}\left[\left[\mathbf{F}_{2,1,1}-\mathbf{F}_{2,1,1}^{*}\right]\left(\mathbf{F}_{2,1,1}^{*}\right)^{-1}+\mathbf{I}\right]\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}\mathbf{e}_j\\ =\left[\mathbf{F}_{2,1,1}-\mathbf{F}_{2,1,1}^{*}\right]^{-1}\left(\mathbf{F}_{2,1,1}^{*}\right)^{-1}\mathbf{M}^{-1}\mathbf{e}_j;\end{aligned} \tag{490}$$

while, for the RHS of condition **II**,

$$\begin{aligned}\mathbf{x}_{4,j}^{*}=\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}\right)^{*}\mathbf{e}_j=\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{*}\right)^{-1}\left(\mathbf{F}_{2,1,1}^{*}\right)^{-1}\mathbf{M}^{-1}\mathbf{e}_j\\ =\left[\mathbf{F}_{2,1,1}-\mathbf{F}_{2,1,1}^{*}\right]^{-1}\left(\mathbf{F}_{2,1,1}^{*}\right)^{-1}\mathbf{M}^{-1}\mathbf{e}_j.\end{aligned} \tag{491}$$

Hence, both conditions are true and

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]=\mathbf{F}_{2,1,1}^{*}\implies\boldsymbol{\gamma}_{2,i}=\boldsymbol{\gamma}_{1,i}^{*}. \tag{492}$$

This result is also consistent with the relation between the integrating constants $\mathbf{C}_1$ and $\mathbf{C}_2$, as proven in Eq. (674), for they are also complex-conjugate when $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$ and $\mathbf{F}_{2,1,1}$ are complex-conjugate.

Putting all two conditions together with Eq. (472) yields

$$\begin{aligned}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]=\mathbf{F}_{2,1,1}^{*}\implies\mathbf{y}_{\mathbf{p}}(t_{i+1})=t_{i+1}\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}\\ -\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}\right)+\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}+\left(\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}\right)^{*},\\ \therefore\mathbf{y}_{\mathbf{p}}(t_{i+1})=t_{i+1}\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}-\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\left(\mathbf{x}_{2,j}+\mathbf{x}_{3,j}\right)\\ +2\Re\left(\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}\right).\end{aligned} \tag{493}$$

Nonetheless, condition **II)** and condition **I)** can both be rearranged to

$$\begin{aligned}\mathbf{x}_{1,j}=\mathbf{x}_{4,j}^{*}+\mathbf{x}_{4,j}=2\Re\left(\mathbf{x}_{4,j}\right),\\ \mathbf{x}_{2,j}+\mathbf{x}_{3,j}=\mathbf{x}_{5,j}^{*}+\mathbf{x}_{5,j}=2\Re\left(\mathbf{x}_{5,j}\right).\end{aligned} \tag{494}$$

Substituting these results into Eq. (493) yields

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]=\mathbf{F}^{*}_{2,1,1} \implies \mathbf{y_p}(t_{i+1})=2t_{i+1}\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\Re\left(\mathbf{x}_{4,j}\right)+2\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,0}\Re\left(\mathbf{x}_{4,j}\right)$$
$$-2\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\Re\left(\mathbf{x}_{5,j}\right)+2\Re\left(\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}\right). \tag{495}$$

If the coefficients $\hat{c}_{j,k,0}$ and $\hat{c}_{j,k,1}$ are real-valued, *i.e.* $\hat{c}_{j,k,0},\hat{c}_{j,k,1}\in\mathbb{R}\ \ \forall j\in\{1,2,\ldots,n_e\},\forall k\in\{0,1,2,\ldots,n_k\}$, then, Eq. (495) can be simplified even further to

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]=\mathbf{F}^{*}_{2,1,1},\ \hat{c}_{j,k,0},\hat{c}_{j,k,1}\in\mathbb{R}\ \forall j\in\{1,2,\ldots,n_e\},\forall k\in\{0,1,2,\ldots,n_k\} \implies$$
$$\mathbf{y_p}(t_{i+1})=t_{i+1}\underbrace{\sum_{k=0}^{i}2\Re\left(\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{4,j}\right)}_{\mathbf{w}_{1,i}}+\underbrace{\sum_{k=0}^{i}2\Re\left(\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{4,j}\right)}_{\mathbf{w}_{2,i}}-\underbrace{\sum_{k=0}^{i}2\Re\left(\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{5,j}\right)}_{\mathbf{w}_{3,i}}$$
$$+2\Re\left(\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}\right). \tag{496}$$

In the same fashion from Eq. (475) and from Eq. (476), let define two new vectors $\mathbf{d}$ using notation from Eq. (480),

$$\mathbf{d}_{8,i+1}=\mathbf{X}_4\begin{bmatrix}\hat{c}_{\mathscr{S}(1),i+1,1}\\\hat{c}_{\mathscr{S}(2),i+1,1}\\\cdots\\\hat{c}_{\mathscr{S}(n_e),i+1,1}\end{bmatrix},$$
$$\mathbf{d}_{9,i+1}=\mathbf{X}_4\begin{bmatrix}\hat{c}_{\mathscr{S}(1),i+1,0}\\\hat{c}_{\mathscr{S}(2),i+1,0}\\\cdots\\\hat{c}_{\mathscr{S}(n_e),i+1,0}\end{bmatrix}, \tag{497}$$

Comparing Eq. (496) to Eq. (475) and to Eq. (476), one realizes that

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]=\mathbf{F}^{*}_{2,1,1},\ \hat{c}_{j,k,0},\hat{c}_{j,k,1}\in\mathbb{R}\ \forall j\in\{1,2,\ldots,n_e\},\forall k\in\{0,1,2,\ldots,n_k\} \implies$$
$$\mathbf{w}_{1,i+1}=\mathbf{w}_{1,i}+2\Re\left(\mathbf{d}_{8,i+1}\right),$$
$$\mathbf{w}_{2,i+1}=\mathbf{w}_{2,i}+2\Re\left(\mathbf{d}_{9,i+1}\right),$$
$$\mathbf{w}_{3,i+1}=\mathbf{w}_{3,i}+2\Re\left(\mathbf{d}_{5,i+1}\right),$$
$$\boldsymbol{\gamma}_{1,i+1}=\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}-t_{i+1}\mathbf{d}_{8,i+1}-\mathbf{d}_{9,i+1}+\mathbf{d}_{5,i+1}. \tag{498}$$

Thus, when $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ and $\mathbf{F}_{2,1,1}$ are complex-conjugate, the number of matrix-vector operations regarding the excited DOFs is 3, in opposition to the 5 required when they are not complex-conjugate. In summary, for overdamped problems, the total number of operations per iterations is: 2 matrix-vector multiplications ($n \times n \cdot n \times 1$) and 5 matrix-vector multiplications ($n \times n_e \cdot n_e \times 1$); whereas, for underdamped problems, 1 matrix-vector multiplication ($n \times n \cdot n \times 1$) and 3 matrix-vector multiplications ($n \times n_e \cdot n_e \times 1$). After all the previous considerations, the calculation of the response in the singularity points t_k can be summarized in Alg. 6

4.3.2 Computational implementation

As it was already stated previously, the most efficient way to compute the response is to set the same set of discrete time points for all the excited DOFs. This permits the use of the same exponential maps, $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)$ and $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)$, in all iterations, despite the different excited DOFs, as can be seen in Alg. 6. For simplicity, the previous equations were derived for equally spaced time points, but that is not necessary for them to work since Δt can be varied within each iteration. This broader approach, with variable time step, was not included to keep notation simple and maintain implementation straightforward. The major computation drawback, however, would be to calculate or to approximate the exponential maps in each or in some iterations.

The update of the vectors $\mathbf{d}$ present a upper bound of computational effort, for they are given by a matrix-vector multiplication. The complexity of such operations are $\mathscr{O}\left(nn_e\right)$, and the upper limit is when $n_e = n$. In many applications, although, it is common to have loads applied only in some regions of the system, such that most of the DOFs are not excited, $n_e \ll n$. In such cases, the operations with the exponential maps dominate the computational effort.

4.3.2.1 Bundles of excited DOFs

In situations where n_e is large relative to n and it impacts on computational effort, alternatives can be suggested to reduce n_e. One such suggestion proposed by the authors is to aggregate some DOFs in bundles, the criterion must be how similar they vary with time. Objectively, one can bundle DOFs up considering frequency and phase for instance. Then, the coefficients $\hat{c}_{j,k,p}$ can be shared among the DOFs contained in the same bundle, which allows to sum the corresponding columns of the $\mathbf{X}_i$ matrices in a pre-processing step, before the iterative procedure. Recollect Eq. (479) and let a set of DOFs with same characteristics, $\mathscr{B}_m$ of cardinality $n_{\mathscr{B},m}$, share the same coefficients $b_{\mathscr{B}_m(q)}$,

$$\sum_{j\in\mathscr{S}} b_{\mathscr{S}(j)}\mathbf{x}_{i,\mathscr{S}(j)} = b_{\mathscr{S}(1)}\mathbf{x}_{i,\mathscr{S}(1)} + b_{\mathscr{S}(2)}\mathbf{x}_{i,\mathscr{S}(2)} + \cdots + b_{\mathscr{B}_m(1)}\mathbf{x}_{i,\mathscr{B}_m(1)} + b_{\mathscr{B}_m(2)}\mathbf{x}_{i,\mathscr{B}_m(2)} + \cdots + b_{\mathscr{B}_m\left(n_{\mathscr{B},m}\right)}\mathbf{x}_{i,\mathscr{B}_m\left(n_{\mathscr{B},m}\right)} + \cdots + b_{\mathscr{S}(n_e)}\mathbf{x}_{i,\mathscr{S}(n_e)}, \tag{499}$$

as the coefficients are equal, $b_{\mathscr{B}_m(1)} = b_{\mathscr{B}_m(2)} = \cdots = b_{\mathscr{B}_m\left(n_{\mathscr{B},m}\right)} = b_{\mathscr{B},m}$,

$$\sum_{j\in\mathscr{S}} b_{\mathscr{S}(j)}\mathbf{x}_{i,\mathscr{S}(j)} = b_{\mathscr{S}(1)}\mathbf{x}_{i,\mathscr{S}(1)} + b_{\mathscr{S}(2)}\mathbf{x}_{i,\mathscr{S}(2)} + \cdots + b_{\mathscr{B},m}\underbrace{\left(\mathbf{x}_{i,\mathscr{B}_m(1)} + \mathbf{x}_{i,\mathscr{B}_m(2)} + \cdots + \mathbf{x}_{i,\mathscr{B}_m\left(n_{\mathscr{B},m}\right)}\right)}_{\mathbf{x}_{i,\mathscr{B},m}} + \cdots + b_{\mathscr{S}(n_e)}\mathbf{x}_{i,\mathscr{S}(n_e)}, \tag{500}$$

that can be rewritten in matrix notation,

$$\sum_{j\in\mathscr{S}} b_{\mathscr{S}(j)}\mathbf{x}_{i,\mathscr{S}(j)} = \underbrace{\begin{bmatrix}\mathbf{x}_{i,\mathscr{S}(1)} & \mathbf{x}_{i,\mathscr{S}(2)} & \cdots & \mathbf{x}_{i,\mathscr{B},m} \cdots & \mathbf{x}_{i,\mathscr{S}(n_e)}\end{bmatrix}}_{\mathbf{X}_i^R}\underbrace{\begin{bmatrix} b_{\mathscr{S}(1)} \\ b_{\mathscr{S}(2)} \\ \cdots \\ b_{\mathscr{B},m} \\ \cdots \\ b_{\mathscr{S}(n_e)}\end{bmatrix}}_{\mathbf{b}^R}, \tag{501}$$

where $\mathbf{X}_i^R$ is the reduced form of matrix $\mathbf{X}_i$, and $\mathbf{b}^R$ is the corresponding reduced version of vector $\mathbf{b}$. It follows directly from Eq. (501) that the matrix $\mathbf{X}_i^R$ has $n_e - n_{\mathscr{B},m} + 1$ columns. The derivations above were developed for a single bundle, $\mathscr{B}_m$, but it can be recursively done for multiple bundles: $\mathscr{B}_1$, $\mathscr{B}_2$, ..., $\mathscr{B}_m$. Each bundle compresses the matrix $\mathbf{X}_i$ by its cardinality minus 1.

Now, one needs a systematic approach to construct such bundles. It follows from Eq. (480) and from Eq. (497) that vectors $\mathbf{d}_{1,i+1}$, $\mathbf{d}_{3,i+1}$, $\mathbf{d}_{5,i+1}$ and $\mathbf{d}_{8,i+1}$ depend solely on coefficients $\hat{c}_{j,i+1,1}$. Consequently, if the DOFs in these vectors are divided in bundles that share the same $\hat{c}_{j,k,1}$, the computation cost of evaluating such vectors $\mathbf{d}$ can be considerably diminished. To do it, let the coefficient $a_{1,j,k}$ from Eq. (434) be picked. Then, an average can be calculated and assigned to all the DOFs belonging to a bundle $\mathscr{B}_m$,

$$\hat{a}_{1,\mathscr{B}_m(1),k} = \hat{a}_{1,\mathscr{B}_m(2),k} = \cdots = \hat{a}_{1,\mathscr{B}_m\left(n_{\mathscr{B},m}\right),k} = \bar{a}_{1,\mathscr{B}_m,k} = \frac{1}{n_{\mathscr{B},m}}\sum_{q=1}^{n_{\mathscr{B},m}} a_{1,\mathscr{B}(q),k}, \tag{502}$$

in which, $a_{1,\mathscr{B}(q),k}$ is the original coefficient $a_{1,j,k}$ for the q-th DOF of the bundle $\mathscr{B}_m$, and $\hat{a}_{1,\mathscr{B}_m(q),k}$ are the new averaged a_1 coefficients. If all the coefficients are equal to each other, it is observed from Eq. (438) that so will be the coefficients $\hat{c}_{j,k,1}$, henceforth, matrices $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_5$ and $\mathbf{X}_4$ can be compressed.

The impact of such proposal in computational cost is readily visible, but the impact on solution accuracy, however, is trickier to assess and depends greatly on how the bundles are created. This investigation is out of the scope of this study, thus, this work will keep with the original and uncompressed matrices in the numerical experiments, and the bundle approach is left as a suggestion for future works.

Algorithm 3: Evaluation of the response $\mathbf{y_p}$ at the Heaviside singularity points $t_{j,k}$ using Eq. (472) or Eq. (493) and the update rules from Eq. (476)

Calculate $\bar{\mathbf{K}}$, $\bar{\mathbf{C}}$, $\mathbf{F}_{2,1,1}$ and $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)$
Initialize a vector $n \times 1$ for $\boldsymbol{\gamma}_1$
Initialize a matrix $n \times n_e$ for each of $\mathbf{X}_4$ and $\mathbf{X}_5$
if $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] = \mathbf{F}^*_{2,1,1}$ **then**
    $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right) = \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)^*$
    Calculate $\boldsymbol{\Omega}_4$ and $\boldsymbol{\Omega}_5$
else
    Initialize a vector $n \times 1$ for $\boldsymbol{\gamma}_2$
    Initialize a matrix $n \times n_e$ for each of $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$
    Calculate $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)$
    Calculate $\boldsymbol{\Omega}_1$, $\boldsymbol{\Omega}_2$, $\boldsymbol{\Omega}_3$, $\boldsymbol{\Omega}_4$ and $\boldsymbol{\Omega}_5$
end
Iterate through the set of excited degrees of freedom $\mathscr{S}$
for $j = 1, 2, \ldots, n_e$
    Calculate $\mathbf{x}_i\left[j\right] = \boldsymbol{\Omega}_i \backslash \mathbf{e}_j$
    Calculate coefficients a_{0j} and a_{1j} using Eq. (436) and Eq. (434)
    Initialize $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ and update $\mathbf{d}_{i,0}$ using Eq. (482)
    Initialize the response as null and sums the initial conditions vectors $\mathbf{C}_1$ and $\mathbf{C}_2$
end
Iterate through the remaining points in time, $\{t_1, t_2, t_3, \ldots, t_{n_k}\}$
for $i = 1, 2, \ldots, n_k$
    if $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] = \mathbf{F}^*_{2,1,1}$ **then**
        Calculate $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_1$
        Calculate the response at time t_i, $\mathbf{y_p}\left(t_i\right)$, using Eq. (496)
    else
        Calculate $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_1$ and $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_2$
        Calculate the response at time t_i, $\mathbf{y_p}\left(t_i\right)$, using Eq. (472)
    end
    Iterate through the set of excited degrees of freedom $\mathscr{S}$
    for $j = 1, 2, \ldots, n_e$
        Calculate coefficients $\hat{c}_{ji0}$ and $\hat{c}_{ji1}$ using Eq. (437) and Eq. (438)
    end
    if $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] = \mathbf{F}^*_{2,1,1}$ **then**
        Update $\mathbf{d}_8$, $\mathbf{d}_9$, $\mathbf{d}_5$, $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ and $\boldsymbol{\gamma}_1$ using Eq. (498).
    else
        Update $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, $\mathbf{d}_4$, $\mathbf{d}_5$, $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ using Eq. (481)
    end
end

4.4 STABILITY ANALYSIS

A measure of stability of a numerical time integration method is an assessment of how error can build up along the iterations and turn the calculated response unbounded (BUTCHER, 2016). To derive expressions for this measure, one can write the response at a time t_{i+1} as a function of previous time points evaluations (BUTCHER, 2016; KAMALI et al., 2023). The result is alike Eq. (2), where there is a matrix $\mathbf{A}$, called the amplification matrix, and its eigenvalues tell how and when such method is *stable*. When the stability does not depend upon the time step, Δt, the method is said to be unconditionally stable and the choice of time step is made according to accuracy only (NOH; BATHE, 2018).

The evaluation of the response in a point t_{i+1} as function of previously evaluated point was already carried out for the efficient evaluation of the Heaviside series method for time integration in Eq. (472). Although the terms $\mathbf{y_p}(t_i)$ does not appear explicitly, it is clear by the recurrence and update relations in Eq. (476) that the response $\mathbf{y_p}(t_{i+1})$ depends upon a local term, given by operations with the vectors $\mathbf{w}$, and upon terms coming from previous iterations, namely $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$. From Eq. (481), it follows that $\mathbf{y_p}(t_{i+1})$ is a function directly of $\mathbf{y_p}(t_i)$, and, thus, the behavior of the operations of this function must be analysed to evaluate stability characteristics.

If matrices $\mathbf{M}$ and $\mathbf{K}$ are non-singular, which is a fair hypothesis for properly applied boundary conditions in FEA, and if $\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]$ is also non-singular, which is explored in Appendix C.1, it is possible to deduce that vectors $\mathbf{x}_{i,j}$ will be bounded, *i.e.* they will have its components bounded, since matrices $\boldsymbol{\Omega}_i$ in Eq. (457), Eq. (459), Eq. (461), Eq. (463) and Eq. (465) are also bounded. It is also possible to reason that, if coefficients $\hat{c}_{j,k,0}$ and $\hat{c}_{j,k,1}$ are bounded, so are the vectors $\mathbf{d}$ in Eq. (480). In fact, as coefficients $\hat{c}_{j,k,0}$ and $\hat{c}_{j,k,1}$ are evaluated using the integral of the function, if the points of integration t_k are chosen such as the integral of the represented function is bounded, so will be the coefficients. Furthermore, if the represented function is bounded, it is straightforward to observe that the integral and, by consequence, the coefficients too will be bounded. With this reasoning, it is clear that the relations in the update rules of Eq. (476) will yield bounded results.

The last operation to analyse is the exponential terms $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)$ and $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)$. For generalization purposes, let evaluate the exponential of a generic matrix $\mathbf{A}$ using the Jordan canonical form (HIGHAM, 2008),

$$\exp\left(-\mathbf{A}\Delta t\right) = \exp\left(-\Delta t \mathbf{Z}^{-1}\mathbf{J}\mathbf{Z}\right) = \mathbf{Z}^{-1}\exp\left(-\Delta t \mathbf{J}\right)\mathbf{Z} = \mathbf{Z}^{-1} diag\left(\exp\left(-\Delta t \mathbf{J}_j\right)\right)\mathbf{Z}, \tag{503}$$

where the exponential of the j-th Jordan block is defined for this problem, according to (HIGHAM, 2008), as

$$\exp\left(-\Delta t \mathbf{J}_j\right) = \begin{bmatrix} \exp\left(-\Delta t \lambda_j\right) & -\Delta t \exp\left(-\Delta t \lambda_j\right) & \dots & -\Delta t^{m_j-1} \frac{\exp\left(-\Delta t \lambda_j\right)}{\left(m_j-1\right)!} \\ 0 & \exp\left(-\Delta t \lambda_j\right) & \ddots & \vdots \\ \vdots & \vdots & \ddots & -\Delta t \exp\left(-\Delta t \lambda_j\right) \\ 0 & 0 & & \exp\left(-\Delta t \lambda_j\right) \end{bmatrix}, \qquad (504)$$

in which, λ_j is the j-th eigenvalue of $\mathbf{A}$ and m_j is its geometric multiplicity. Thus, one can easily reason that, if all eigenvalues of $\mathbf{A}$ are strictly positive, the exponential in Eq. (503) will decrease and converge to a null matrix as Δt increases, since the exponential of the Jordan form will decay. Therefore, if the real part of the eigenvalues of both $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ are strictly positive, $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)$ and $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)$ will converge to the null matrix as Δt increases, rendering these matrices bounded for positive Δt. As all terms in Eq. (472) and in Eq. (481) are bounded for stable systems (with no negative damping), it is possible to assure that the method is unconditionally stable when the eigenvalues of $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ are positive, as the exponential maps are bounded too.

The exponential maps of $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ also appear in the analytical homogeneous solutions, Equation (441). Thus, if the eigenvalues of $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ have negative real part, the analytical homogeneous response will not be bounded. This result shows that there are conditions regarding Rayleigh proportional damping for the analytical response to be bounded. As a matter of fact, Equation (474) and Equation (482) show that the proposed method acts as a modulator for the homogeneous response, *i.e.*, it approximates the loading and uses the homogeneous response to approximate the particular solution while keeping the latter exact. Thus, when the eigenvalues of $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ have negative real part, it is not the HS method that is unstable, but the physical system itself. For this reason, Appendix C.3 presents conditions, Eq. (744), for Rayleigh damping to produce physically stable solutions, both analytically and numerically.

It is worth noting that Appendix C.3 can be easily extended for generalized proportional damping, like in (ADHIKARI, 2006), since matrices in that work are simultaneously diagonalizable. Nonetheless, if $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ are not simultaneously diagonalizable, there is a basis where both commuting matrices are triangular when they commute (HORN; JOHNSON, 2012), and it is well known that the eigenvalues of triangular matrices are their main diagonal entries. Henceforth, calculations can be made for these more general cases of damping in an analogous fashion to Appendix C.3.

4.5 RESULTS

A simple benchmark problem is proposed to assess the proposed formulation. The problem is a 3 DOFs system described by

$$\mathbf{M} = \begin{bmatrix} 5.0 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 3.0 \end{bmatrix}, \tag{505}$$

$$\mathbf{K} = \begin{bmatrix} 6.0 & -4.0 & 0.0 \\ -4.0 & 10.0 & -3.0 \\ 0.0 & -3.0 & 7.0 \end{bmatrix} \times 10^2 \tag{506}$$

and

$$\mathbf{C} = \beta \mathbf{K}, \tag{507}$$

with

$$\mathbf{y}(0) = \dot{\mathbf{y}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{508}$$

where the damping parameter is $\beta = 1 \times 10^{-6}$. The time span is $t \in [0, 10]s$ in all simulations.

This system has 3 natural frequencies: 1.352 Hz, 2.266 Hz and 3.828 Hz. Thus, the smaller natural period of the response is 0.26 s. This system is subjected to a simple sinusoidal excitation in the second DOF with unitary amplitude and different excitation frequencies, according to Figure 17 and to Figure 18. This excitation was chosen since periodic continuous function can be approximated by a linear combination of sines by means of Fourier Series (KREYSZIG; KREYSZIG; NORMINTON, 2011).

The influence of the time step, Δt, is also studied for each excitation frequency. The Root Mean Squared Mean Error (RMSE), (JAMES et al., 2013), of the complete solution obtained by using the proposed approach and also obtained by using the Newmark-beta method are evaluated using the analytical solution from Chapter 3 as reference. The excitation frequency also impacts the accuracy of numerical methods, due to problems like aliasing. Consequently, a wide range of frequencies must be tested, which is observed in Fig. 17 and in Fig. 18, where each set of circles with the same color represent results with the same excitation frequency for different time steps Δt.

Figure 17 shows that, the smaller the step size Δt, the more accurate is the solution for a given excitation frequency. This improvement in accuracy has a linear tendency in the $\log_{10} \times \log_{10}$ plot, indicating that the proposed approach has a stable rate of convergence regardless of the step size in the linear zone. It can be also seen that this behavior happens for all frequencies, including excitation frequencies near the natural frequencies (as the damping factor is very small, one may assume that the resonance frequencies are very close to the fundamental frequencies).

The same experiment was performed using the Newmark-beta method, as depicted in Fig. 18. This graphic shows that the Newmark method is more affected by excitation frequencies near the natural frequencies when compared to the proposed approach. Also, the linear tendency in the log $\times$ log plot is only achieved for very small time steps, which can be confirmed by the linear region being shifted further to the left in Fig. 18.

The presence of linear zones of error in the $\log_{10} \times \log_{10}$ indicates that a clear relation between error and time discretization can be derived. Let the linear relation between error and time discretization be written in the log scale,

$$\log_{10} RMSE = a \log_{10} \Delta t + b, \tag{509}$$

which can be converted to a exponential equation of base 10 by raising 10 by each side of Eq. (509),

$$RMSE = 10^{a \log_{10} \Delta t + b} = 10^{\log_{10} \Delta t^a} 10^b, \tag{510}$$

that is simplified to

$$RMSE = \Delta t^a 10^b. \tag{511}$$

Using Eq. (511), it is possible to notice that the proposed method reached a convergence order between 4 (quartic) and of 2 (quadratic) depending on excitation frequency, while the Newmark-beta method presented convergence orders between 2 and 1, also depending on the excitation frequency.

4.5.1 Elapsed time

The elapsed time during execution of each method is a strong criterion to choose among numerical methods. An estimate of computation time is possible to do with complexity analysis. In the case of Newmark-beta, for instance, and other numerical methods, it is important to highlight the need for a finer time discretization, to reduce the propagation of errors. Despite that, it is also important to stress that in the implementation of the Newmark-beta, (HUGHES, 2000; LINDFIELD; PENNY, 2019), there are at least 3 matrix-vector multiplications per time step ($\mathcal{O}\left(n^2\right)$, (JIN; CHEW, 2005), where n is the dimensionality of the problem) and the solution of a linear system. This linear system can be pre-factorized such that a forward and a backward substitutions are needed at each time step (($\mathcal{O}\left(n^2\right)$ (FORD, 2015)). Hence, it is estimated that each iteration of the Newmark-beta method has a complexity of $5\mathcal{O}\left(n^2\right)$.

The proposed approach, Alg. 6, requires two matrix-vector multiplications when the problem is over damped, with complexity $2\mathcal{O}\left(n^2\right)$, or just a single matrix-vector product

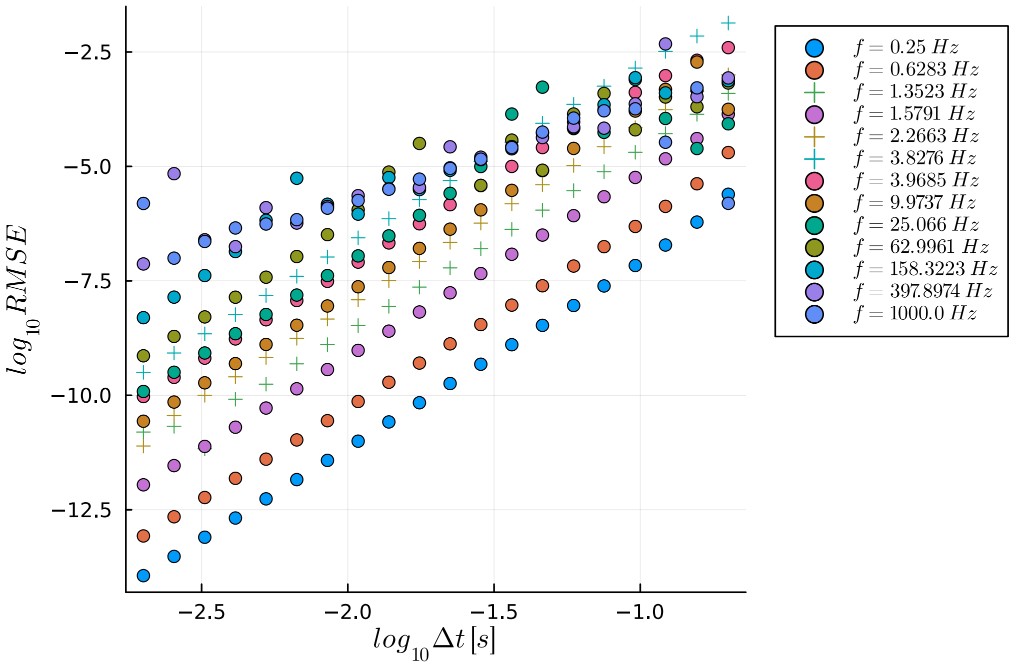

Figure 17 – $\log_{10}$ of the Root Mean Square Error (RMSE) of $\mathbf{y}(t)$ obtained with the proposed approach regarding the time step Δt and the excitation frequency f. Crosses are associated to the fundamental frequencies of the system.

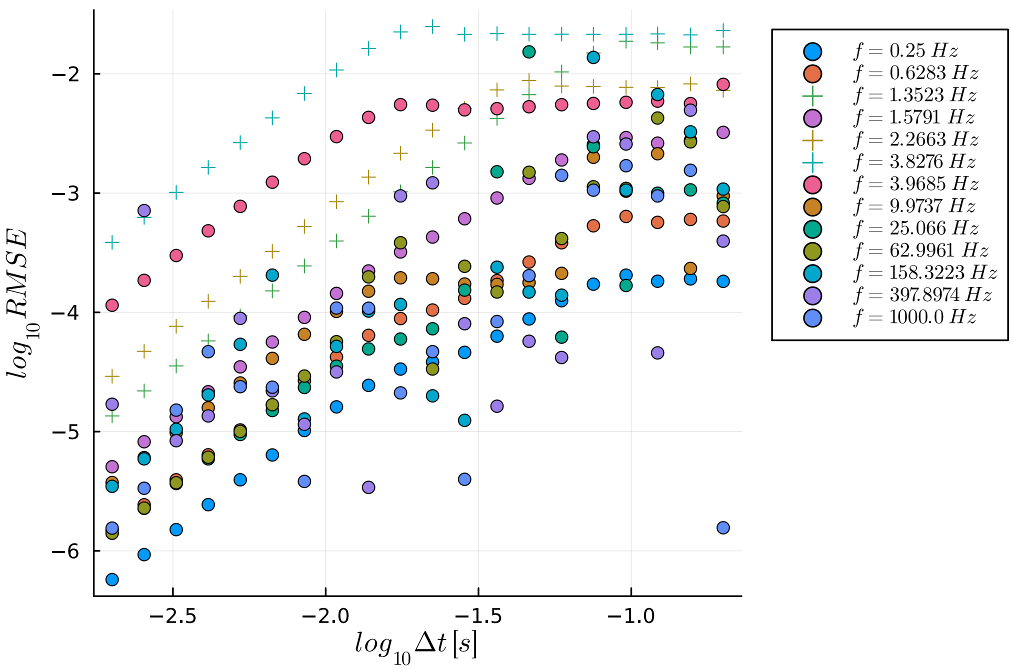

Figure 18 – $\log_{10}$ of the Root Mean Square Error (RMSE) of $\mathbf{y}(t)$ obtained with the Newmark-beta method regarding the time step Δt and the excitation frequency f. Crosses are associated to the fundamental frequencies of the system.

with complexity $\mathscr{O}\left(n^2\right)$ if the system is under damped. The Heaviside series method presents the cost of evaluating the vectors **d** in Eq. (481) and (498). Nevertheless, they are matrix-vector multiplications ($n \times n_e \cdot n \times 1$), so their order of complexity is $\mathscr{O}\left(nn_e\right)$ and, consequently, inferior to $\mathscr{O}\left(n^2\right)$, as $n_e \leq n$. For instance, in underdamped problems, even if $n_e = n$, the complexity per iteration is $4\mathscr{O}\left(n^2\right)$, which is lesser than Newmark-beta's. In addition to the smaller computational complexity, one must also note that the proposed approach is more accurate for a given Δt when compared to the numerical approach. Thus, one can use larger Δt for the same RMSE, reducing the computational effort even more.

A finite element model with increasing number of DOFs is used to investigate the solution time for both the proposed approach and for the Newmark-beta method. The simulation for each time discretization was repeated 13 times and the mean elapsed time and its standard deviation were recorded. All the values were normalized by the maximum mean elapsed time of all simulations. The 13 independent runs were used to take into account inherent variations in elapsed time, associated to compilation times in the Julia language, operational system issues, among other factors.

Figure 19 and Figure 20 show the mean elapsed time and the standard deviation normalized by the maximum time as a function of the number of discrete time steps, n_k. Figure 19 presents this simulation for a structure with 458 DOFs, while Figure 20 for a structure with 1718 DOFs. The mean elapsed time of the Newmark-beta method is given by the blue line and the mean elapsed time of HS is given by the the orange line.

Two features are common to both Figure 19 and Figure 20: the (almost) linearity of all curves with respect to n_k and the intersection of both curves at the beginning of the graph, *i.e.* with low n_k. The linearity was expected, since the total cost is the sum of the cost of each individual iteration, plus some time associated to pre-processing. As the pre-processing of the HS is more expensive, due to solving Eq. (444) and evaluating the matrix exponential, the cost of HS is higher for low n_k. Nonetheless, this cost rapidly decreases relative to the the Newmark-beta method as n_k increases. One can also observe that the cost of the HS method per iteration is smaller than the cost per iteration of the Newmark-beta method, whose immediate consequence is the dilution of the initial cost of this pre-processing. Even more interestingly, comparing Figure 19 to Figure 20 shows that the rate of increase of mean elapsed time of HS method with respect to n_k diminishes comparatively to the rate of the Newmark-beta as the problem dimensionality increases.

Figure 21 and Figure 22 show the effective time to compute the complete solution for different number of DOFs, for both the proposed approach and for the Newmark-beta method. Two time discretizations were chosen to investigate the elapsed time as a function of the number of DOFs: $log_{10}\left(\Delta t\right) = -1.3096$, or $n_k = 103$, and $log_{10}\left(\Delta t\right) = -3$, or $n_k = 5001$. These time steps were chosen since for lower n_k there is a tendency for the Newmark-beta to be more efficient than the proposed approach, whereas the converse is true for large n_k.

Figure 21 shows that the proposed approach is less efficient than the Newmark-beta

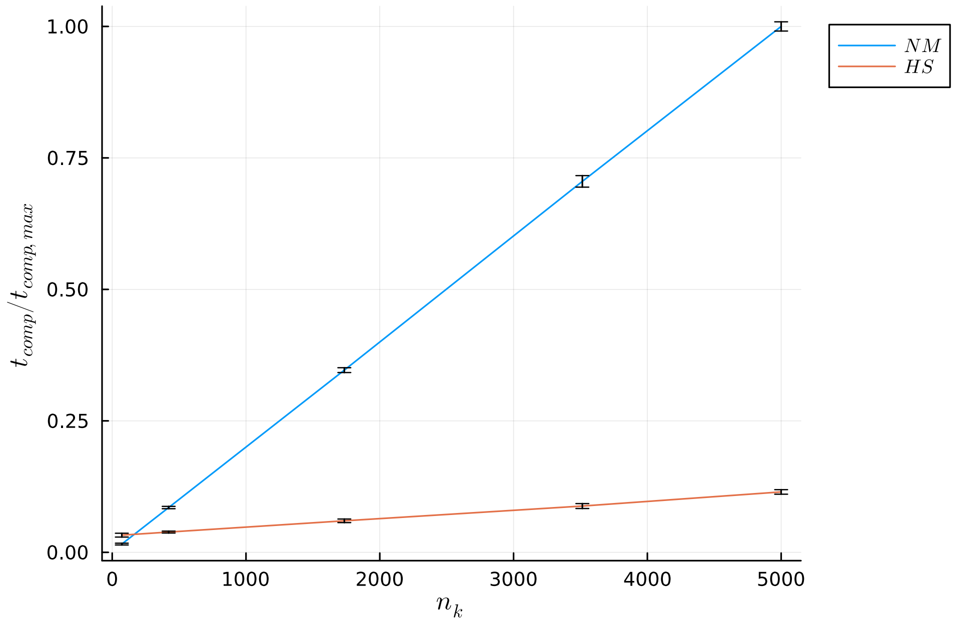

Figure 19 – Normalized mean elapsed time as a function of the number of discrete time steps for an 458 DOFs problem. The blue line refers to the Newmark-beta method and the orange line to the proposed approach. The black error bars represent the standard deviation relative to 13 independent runs.

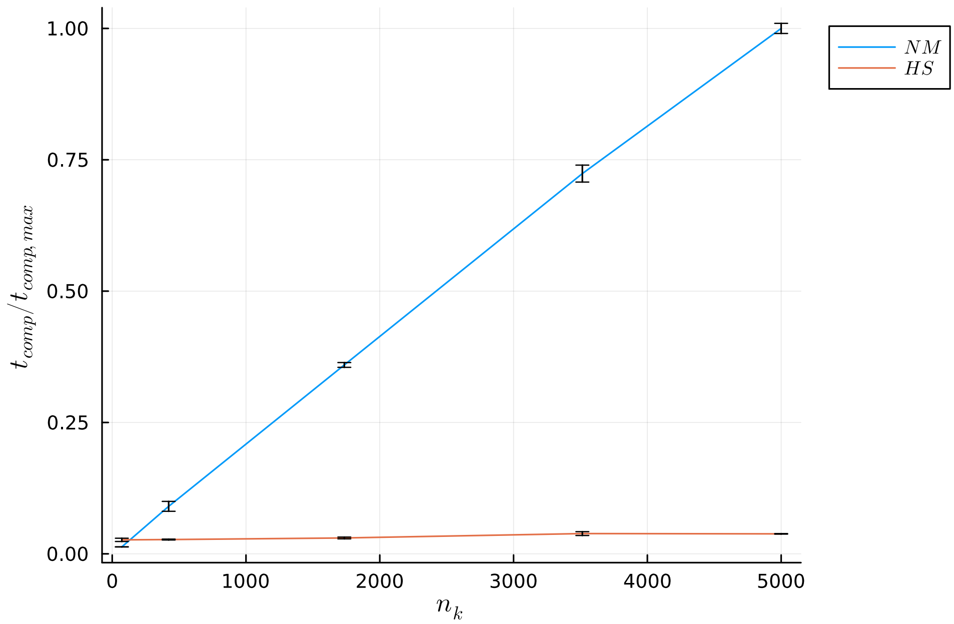

Figure 20 – Normalized mean elapsed time as a function of the number of discrete time steps for an 1718 DOFs problem. The blue line refers to the Newmark-beta method and the orange line to the proposed approach. The black error bars represent the standard deviation relative to 13 independent runs.

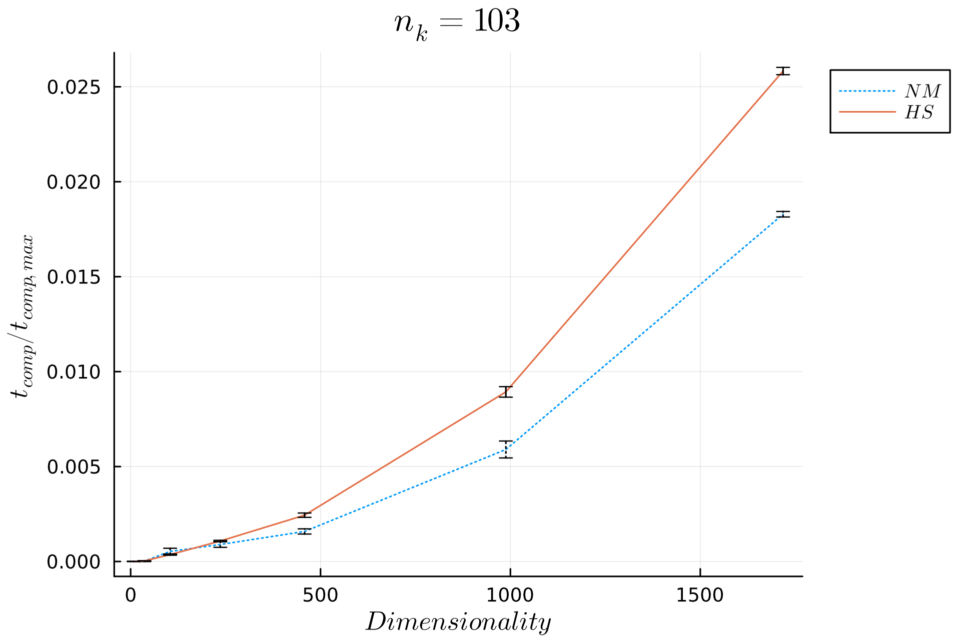

Figure 21 – Normalized mean elapsed time as a function of the dimensionality of the problem, n. The blue line refers to the Newmark-beta method and the orange line to the proposed approach. The black error bars represent the standard deviation relative to 13 independent runs

method for coarse time discretization, disregarding the number of DOFs (one must keep in mind the large difference in accuracy, as shown in previous results). This difference is associated to the pre-processing phase of the proposed approach. Nonetheless, as depicted in Fig. 22, there is a significant difference in performance when the number of discrete time points and the number of DOFs increase.

It is important to note the small deviation in the results (represented by the black error bars) and, especially, by the fact that the trust regions of both methods do not overlap in the majority of the points in the previous graphics.

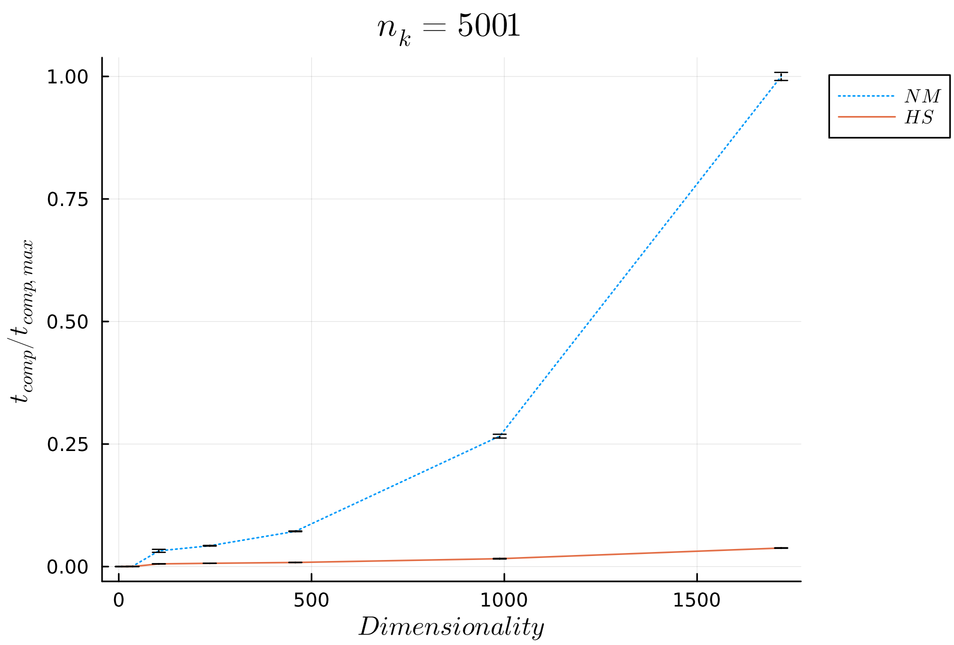

Figure 22 – Normalized mean elapsed time as a function of the dimensionality of the problem, n. The blue line refers to the Newmark-beta method and the orange line to the proposed approach. The black error bars represent the standard deviation relative to 13 independent runs.

4.6 FINAL REMARKS OF THE CHAPTER

This chapter proposed a new approximation technique, which represents a function as a finite sum of Heaviside step functions multiplied by polynomial terms. Despite the relevance of such approximation in its own right, this approach was used to sought for semi-analytical solutions to systems of coupled linear ODEs using the GIF. The solutions are semi-analytical because the GIF yields analytical solutions for excitation functions comprised of series of Heaviside steps, but the excitation itself is not exact, rather an approximation.

The approximation of the load is devised to guarantee that the approximated load has the same integral as the original load in the points where the Heaviside steps are applied, thereby, preserving physical quantity such as impulse. Besides, it is shown that the preservation of the integral of the load contributes to the accuracy of the integral of the response itself, hence, improving the accuracy of integral measures, like the ones used in optimization problems.

The order of approximation is set and shown up to the second order. The implication of the computational cost of each approximation order is demonstrated, and it is shown that the first order approximation is the most cost-effective. For this reason, Alg. 6 was derived to efficiently evaluate first order approximations when the times where the Heaviside steps are applied are uniformly distributed along the simulated time span. Variations of the method were also suggested for future work, above all to reduce the computational burden and make the technique even cheaper. Using the derivation of Alg. 6, the stability analysis of the method could also be derived.

It was shown that the method is unconditionally stable when the eigenvalues of the matrices $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ have non-negative real part. With support of Appendix C.3, it was shown that for some conditions, Rayleigh proportional damping yields eigenvalues with strictly non-negative real part. These conditions, nevertheless, are not a limitation of the proposed approach, but physical aspects of the system, since the exponential maps in the analytical homogeneous solution are bounded only if the eigenvalues have non-negative real part. Henceforth, if the system is physically unstable, the HS method does not mask it with artificial numerical damping.

To assess accuracy and computational cost, the HS method is compared with the analytical solution due to the Generalized Integrating Factor method and with the Newmark-beta method. Different excitation frequencies are tested with different time discretization for FEM models with different number of DOFs. Thus, effects of time discretization and of the problem's dimensionality were evaluated in a set of numerical experiments.

Finally, it was shown that the proposed method yields greater accuracy with less computational effort when compared to the Newmark-beta method. This result makes the HS method a good choice to solve linear systems of coupled ordinary differential equations, for its cost-benefit ratio is considerably larger.

5 NON-CLASSICAL NORMAL MODES

In Chapter 3 and in Chapter 4, the derivation of the closed-form solutions were all made using the assumption that $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute, which is true to undamped systems and to proportional damping, whose application is widespread. Nevertheless, one might reasonably question whether the proposed methods, GIF and HS, are limited to such case and to the simplifications that follow thereafter. The answer is **no** and this chapter will evaluate again many important results to prove it and to show how the solution behaviors when the system does not have classical normal modes.

The focus of this chapter will be on three main results: homogeneous solution, particular solution due to Dirac's delta, and particular solution due to Heaviside step functions multiplied by polynomial coefficients. These three situations were chosen for their importance and for their ability to construct other and more complicated excitation functions. Another justification for this choice is the similar mathematical structure shared by all of the three solutions.

It is proven that the convolution leads to an accessory Sylvester equation, which can be solved once as a preprocessing step. The computation cost is shown not to increase dramatically, and suggestions are made to reduce it. Numerical experiments were not performed, for the application of non-classical normal modes is not common and because the main objective is to extend the mathematical formulation for this case. It is demonstrated, however, for all the presented results, that the solutions found previously are special cases, as they must be.

5.1 LINEAR SECOND ORDER SYSTEMS OF ODES WITH CONSTANT MATRIX COEFFICIENTS AND GENERAL MODES

Systems of coupled second order ODEs with constant matrix coefficients are represented by Eq. (1). As matrix coefficient $\mathbf{M}$ is non-singular, it its possible to rewrite this Equation as

$$\mathbf{I}\ddot{\mathbf{y}}(t)+\bar{\mathbf{C}}\dot{\mathbf{y}}(t)+\bar{\mathbf{K}}\mathbf{y}(t)=\bar{\mathbf{f}}(t) \tag{512}$$

where $\bar{\mathbf{C}}=\mathbf{M}^{-1}\mathbf{C}$, $\bar{\mathbf{K}}=\mathbf{M}^{-1}\mathbf{K}$ and $\bar{\mathbf{f}}(t)=\mathbf{M}^{-1}\mathbf{f}$(t). The Generalized Integrating Factor, an analytical approach to solve this problem, was extended to systems of ODEs in Chapter 3. The general solution for ODEs with constant coefficients is given by

$$\mathbf{y}(t)=\underbrace{\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\int\exp\left(\mathbf{F}_{2,1,1}t\right)\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t}_{\mathbf{y_p}(t)}+$$
$$\underbrace{\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\mathbf{C}_2+\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\mathbf{C}_1}_{\mathbf{y_h}(t)}, \tag{513}$$

where $\mathbf{y_p}(t)$ is the particular solution due to the excitation vector $\mathbf{f}(t)$, $\mathbf{y_h}(t)$ is the homogeneous solution and $\mathbf{C}_1$ and $\mathbf{C}_2$ are vectors of integration constants. The matrix $\mathbf{F}_{2,1,1}$ is the solution to the following matrix equation

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{0}. \tag{514}$$

If the original system has classical normal modes, one observes that $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute (ADHIKARI, 2006), thus, the solution is, according to (HIGHAM, 2008), given in closed form as

$$\mathbf{F}_{2,1,1} = \frac{1}{2}\left[\bar{\mathbf{C}} + \sqrt{\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}}\right]. \tag{515}$$

However, when the system does not have classical normal modes, Equation (514) must be solved numerically and there is, indeed, a large literature on the topic of solving quadratic matrix equations numerically, such as (HIGHAM; KIM, 2001; KIM; HIGHAM; KIM, 2001; LU; AHMED; GUAN, 2016; KIM, 2007; HERNáNDEZ-VERóN; ROMERO, 2019; POLONI, 2011; HASHEMI; DEHGHAN, 2010). Many of these works achieved accurate solvents with low computational time.

As the solution is not necessarily given by Eq. (515) anymore, there is no guarantee that $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}}$ commute, as it was proven in Chapter 3. Therefore, the solution provided in Eq. (513) must be tackled without the simplifications that the commutativity provides. The exploration of such solutions will be carried out step by step, first for the homogeneous solution, then for excitation due to Dirac's delta and, finally, for excitation due to Heaviside step function multiplied by first order polynomials, *i.e.* first order Heaviside Series (HS), as proposed in Chapter 4.

5.1.1 Homogeneous solution

From Eq. (513), the homogeneous solution is given by

$$\mathbf{y_h}(t) = \underbrace{\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t}_{\mathbf{Y_{h,1}}}\,\mathbf{C}_2 + \underbrace{\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)}_{\mathbf{Y_{h,2}}}\mathbf{C}_1 \tag{516}$$

where $\mathbf{Y_{h,1}}$ and $\mathbf{Y_{h,2}}$ are time dependent $n \times n$ matrices. The $\mathbf{Y_{h,2}}$ term is straightforward to compute, as it can be evaluated quite efficiently using discrete time points, as shown in Chapter 3 and Chapter 4. The question lies in term $\mathbf{Y_{h,1}}$, specially regarding the integration. This integral, nonetheless, can be evaluated using integration by parts

$$\begin{aligned}
&\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t \\
&= \int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left(-\dot{\exp}\left(-\mathbf{F_{2,1,1}t}\right)\mathbf{F_{2,1,1}^{-1}}\right)\mathrm{d}t \\
&= -\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}^{-1}+ \\
&\int \left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\mathbf{F}_{2,1,1}^{-1} \\
&= -\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}^{-1} \\
&+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\mathbf{F}_{2,1,1}^{-1};
\end{aligned} \tag{517}$$

multiplying both sides of the equation to the right by $-\mathbf{F}_{2,1,1}$ and rearranging it yields

$$\begin{aligned}
&\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t \\
&-\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\mathbf{F}_{2,1,1} = \\
&\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right).
\end{aligned} \tag{518}$$

In Eq. (516), one observes that the term $\mathbf{Y_{h,1}}$ contains the exponential of $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$. For this reason, Eq. (518) is multiplied to the right by this exponential,

$$\begin{aligned}
&\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t \\
&-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\mathbf{F}_{2,1,1} = \\
&\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right);
\end{aligned} \tag{519}$$

Using the Taylor series definition of the matrix exponential, (HIGHAM, 2008), it is easy to observe that the exponential of a matrix commutes with the matrix itself. Besides, as a matrix commutes with itself, the multiplication of the exponential of a matrix by the exponential of this matrix multiplied by (-1) yields the identity matrix. Consequently, Equation (519) can be simplified to

$$\begin{aligned}
&\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t \\
&-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\mathbf{F}_{2,1,1} = \exp\left(-\mathbf{F}_{2,1,1}t\right).
\end{aligned} \tag{520}$$

Equation (520), however, can be multiplied to the right by the exponential of $\mathbf{F}_{2,1,1}$, which results in

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\exp\left(\mathbf{F}_{2,1,1}t\right)$$
$$-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\mathbf{F}_{2,1,1}\exp\left(\mathbf{F}_{2,1,1}t\right)=\mathbf{I}, \tag{521}$$

or, simply,

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\underbrace{\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\exp\left(\mathbf{F}_{2,1,1}t\right)}_{\mathbf{X}}$$
$$-\underbrace{\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\exp\left(\mathbf{F}_{2,1,1}t\right)}_{\mathbf{X}}\mathbf{F}_{2,1,1}=\mathbf{I}, \tag{522}$$

which can be rewritten as

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\mathbf{X}-\mathbf{X}\mathbf{F}_{2,1,1}=\mathbf{I}, \tag{523}$$

a Sylvester equation with unique solution if and only if $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$ and $\mathbf{F}_{2,1,1}$ do not share any eigenvalue (BARTELS; STEWART, 1972). In Eq. (522), it was noted that the term $\mathbf{X}$ is a function of time t, but, as it must satisfy the Sylvester equation with constant coefficients in Eq. (523), which has a unique solution if there are no common eigenvalue, the time dependency can be omitted and, more importantly, the term $\mathbf{Y_{h,1}}$ can be rewritten as

$$\begin{aligned}\mathbf{Y_{h,1}}(t)&=\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\\&=\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t\right);\end{aligned} \tag{524}$$

and the integral itself can be expressed as

$$\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t=\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t\right), \tag{525}$$

that is easily verified if the LHS is differentiated w.r.t. t and Eq. (523) is applied, which yields the integrand of the RHS.

Finally, the homogeneous solution can be rewritten as follows using Eq. (524),

$$\mathbf{y_h}(t)=\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2+\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\mathbf{C}_1, \tag{526}$$

whose time derivative is

$$\dot{\mathbf{y}}_{\mathbf{h}}(t) = -\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}\mathbf{C}_2 - \exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\mathbf{C}_1. \tag{527}$$

Thus, at time $t = 0$, the initial conditions $\mathbf{u} = \mathbf{y}(0)$ and $\mathbf{v} = \dot{\mathbf{y}}(0)$ are given by

$$\begin{aligned} \mathbf{u} &= \mathbf{y}_{\mathbf{p}}(0) + \mathbf{X}\mathbf{C}_2 + \mathbf{C}_1, \\ \mathbf{v} &= \dot{\mathbf{y}}_{\mathbf{p}}(0) - \mathbf{X}\mathbf{F}_{2,1,1}\mathbf{C}_2 - \left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\mathbf{C}_1, \end{aligned} \tag{528}$$

which can be rewritten in a system of equations,

$$\begin{bmatrix} \mathbf{I} & \mathbf{X} \\ -\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right] & -\mathbf{X}\mathbf{F}_{2,1,1} \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}-\mathbf{y}_{\mathbf{p}}(0) \\ \mathbf{v}-\dot{\mathbf{y}}_{\mathbf{p}}(0) \end{bmatrix}. \tag{529}$$

By multiplying the first row to the left by $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$ and adding it to the second row, one gets

$$\begin{bmatrix} \mathbf{I} & \mathbf{X} \\ \mathbf{0} & \left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\mathbf{X}-\mathbf{X}\mathbf{F}_{2,1,1} \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}-\mathbf{y}_{\mathbf{p}}(0) \\ \mathbf{v}-\dot{\mathbf{y}}_{\mathbf{p}}(0)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](\mathbf{u}-\mathbf{y}_{\mathbf{p}}(0)) \end{bmatrix}. \tag{530}$$

Inspecting the term at the second row and second column of Eq. (530) and comparing it to Eq. (523), one easily realizes that the system can be further simplified to

$$\begin{bmatrix} \mathbf{I} & \mathbf{X} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}-\mathbf{y}_{\mathbf{p}}(0) \\ \mathbf{v}-\dot{\mathbf{y}}_{\mathbf{p}}(0)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](\mathbf{u}-\mathbf{y}_{\mathbf{p}}(0)) \end{bmatrix}, \tag{531}$$

whose solution is

$$\begin{aligned} \mathbf{C}_2 &= \mathbf{v}-\dot{\mathbf{y}}_{\mathbf{p}}(0)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](\mathbf{u}-\mathbf{y}_{\mathbf{p}}(0)), \\ \mathbf{C}_1 &= \mathbf{u}-\mathbf{y}_{\mathbf{p}}(0)-\mathbf{X}\mathbf{C}_2. \end{aligned} \tag{532}$$

5.1.1.1 Homogeneous solution when the modes are classical normal

As proven in Appendix D.1, when the system has classical normal modes, the Sylvester equation has a closed form solution. If this solution is substituted in Eq. (532) and, then, in Eq. (526), one gets

$$\begin{aligned} \mathbf{y}_{\mathbf{h}}(t) = {} & \left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\exp\left(-\mathbf{F}_{2,1,1}t\right)\left[\mathbf{v}-\dot{\mathbf{y}}_{\mathbf{p}}(0)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](\mathbf{u}-\mathbf{y}_{\mathbf{p}}(0))\right] \\ & +\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\Big[\mathbf{u}-\mathbf{y}_{\mathbf{p}}(0)-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\left[\mathbf{v}-\dot{\mathbf{y}}_{\mathbf{p}}(0)\right. \\ & \left.+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](\mathbf{u}-\mathbf{y}_{\mathbf{p}}(0))\right]\Big], \end{aligned} \tag{533}$$

which is further simplified using the commutativity of $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ proven in Chapter 3,

$$\begin{aligned}\mathbf{y_h}(t) = \exp\left(-\mathbf{F}_{2,1,1}t\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\left[\mathbf{v}-\dot{\mathbf{y}}_{\mathbf{p}}(0)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](\mathbf{u}-\mathbf{y_p}(0))\right] \\ +\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\Big[\mathbf{u}-\mathbf{y_p}(0)-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\left[\mathbf{v}-\dot{\mathbf{y}}_{\mathbf{p}}(0)\right. \\ \left.+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](\mathbf{u}-\mathbf{y_p}(0))\right]\Big], \qquad (534)\end{aligned}$$

that is equal to the homogeneous response derived in Chapter 3. Thus, the expression proposed in Eq. (526) is consistent to classical normal modes, as it should be.

5.1.2 Particular solution due to Dirac's delta excitation

From Eq. (513), the particular solution due to excitation is given by

$$\mathbf{y_p}(t) = \exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\int \exp\left(\mathbf{F}_{2,1,1}t\right)\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t. \qquad (535)$$

Let the excitation vector be particularized to

$$\mathbf{f}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} c_{jk}\delta(t-t_k)\mathbf{e}_j, \qquad (536)$$

where c_{jk} is a constant, $\delta\left(t-t_{jk}\right)$, the Dirac's delta at $t=t_{jk}$, n_k is the number of pulses, and $\mathbf{e}_j=\{\delta_{ij}\}, i=\{1,2,\ldots,n\}$, where δ_{ij} is the Kronecker's delta. Thus, the normalized excitation vector is given by

$$\bar{\mathbf{f}}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} c_{jk}\delta(t-t_k)\mathbf{M}^{-1}\mathbf{e}_j = \sum_{k=0}^{n_k}\sum_{j=1}^{n} c_{jk}\delta(t-t_k)\mathbf{v}_j. \qquad (537)$$

Applying this excitation to Eq. (535) and using the linearity property of the integral operator, the particular solution is given by

$$\begin{aligned}\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} c_{jk}\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right) \\ \int \exp\left(\mathbf{F}_{2,1,1}t\right)\delta(t-t_k)\,\mathrm{d}t\,\mathrm{d}t\,\mathbf{v}_j, \qquad (538)\end{aligned}$$

whose inner convolution can be evaluated using the Dirac's delta filter property, (KANWAL, 2011),

$$\begin{aligned}\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} c_{jk}\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right) \\ \exp\left(\mathbf{F}_{2,1,1}t_k\right)\mathscr{H}(t-t_k)\,\mathrm{d}t\,\mathbf{v}_j, \qquad (539)\end{aligned}$$

where $\mathscr{H}(t-t_k)$ is the Heaviside step function with discontinuity in $t=t_k$.

The constant matrix exponential that resulted from the inner convolution in Eq. (539) can be put out of the outer convolution and its integration limits can be changed using the property of the Heaviside step functions discussed in Chapter 3,

$$\mathbf{y_p}(t)=\sum_{k=0}^{n_k}\sum_{j=1}^{n}c_{jk}\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int_{t_k}^{t}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t$$
$$\exp\left(\mathbf{F}_{2,1,1}t_k\right)\mathscr{H}(t-t_k)\,\mathbf{v}_j. \qquad (540)$$

The integral term in Eq. (540) was already developed in Eq. (525), hence, it can be readily used here,

$$\mathbf{y_p}(t)=\sum_{k=0}^{n_k}\sum_{j=1}^{n}c_{jk}\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left[\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t\right)-\right.$$
$$\left.\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t_k\right)\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t_k\right)\right]\exp\left(\mathbf{F}_{2,1,1}t_k\right)\mathscr{H}(t-t_k)\,\mathbf{v}_j. \qquad (541)$$

which further simplifies to

$$\mathbf{y_p}(t)=\sum_{k=0}^{n_k}\sum_{j=1}^{n}c_{jk}\left[\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k-t)\right)-\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k-t)\right)\mathbf{X}\right]\mathscr{H}(t-t_k)\,\mathbf{v}_j. \qquad (542)$$

5.1.2.1 Solution due to Dirac's delta when the modes are classical normal

It was shown in Appendix D.1 that, when the system has classical normal modes, the solution of the Sylvester equation, $\mathbf{X}$, commutes with $\mathbf{F}_{2,1,1}$, so Equation (542) can be rearranged to

$$\mathbf{y_p}(t)=\sum_{k=0}^{n_k}\sum_{j=1}^{n}c_{jk}\left[\exp\left(\mathbf{F}_{2,1,1}(t_k-t)\right)\mathbf{X}-\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k-t)\right)\mathbf{X}\right]\mathscr{H}(t-t_k)\,\mathbf{v}_j, \qquad (543)$$

which is simplified to

$$\mathbf{y_p}(t)=\sum_{k=0}^{n_k}\sum_{j=1}^{n}c_{jk}\left[\exp\left(\mathbf{F}_{2,1,1}(t_k-t)\right)-\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k-t)\right)\right]\mathbf{X}\mathscr{H}(t-t_k)\,\mathbf{v}_j, \qquad (544)$$

and, substituting the solution $\mathbf{X}$ from Appendix D.1 into Eq. (544), yields

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} c_{jk}\left[\exp\left(\mathbf{F}_{2,1,1}(t_k - t)\right) - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t)\right)\right]$$
$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{v}_j\mathscr{H}(t - t_k), \tag{545}$$

which is exactly the same found in Chapter 3. Thereby, Equation (542) is consistent with classical normal modes.

5.1.3 Particular solution due to first order HS

Let the excitation vector be given by

$$\mathbf{f}(t) = \sum_{j=1}^{n}\sum_{k=0}^{n_k}\left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t\right)\mathscr{H}(t - t_k)\,\mathbf{e}_j \tag{546}$$

where $\hat{c}_{j,k,0}$ and $\hat{c}_{j,k,1}$ are constant coefficients. Normalized by the mass matrix, it becomes

$$\bar{\mathbf{f}}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n}\left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t\right)\mathscr{H}(t - t_k)\,\mathbf{v}_j; \tag{547}$$

applying it to Eq. (535) and using the linearity of the integral operator, the particular response is given by

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n}\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)$$
$$\int\exp\left(\mathbf{F}_{2,1,1}t\right)\left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t\right)\mathscr{H}(t - t_k)\,\mathrm{d}t\,\mathrm{d}t\,\mathbf{v}_j. \tag{548}$$

The property of change of integration limits of the Heaviside step function can be used, yielding

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n}\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\int_{t_k}^{t}\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)$$
$$\int_{t_k}^{t}\exp\left(\mathbf{F}_{2,1,1}t\right)\left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t\right)\mathrm{d}t\,\mathrm{d}t\,\mathscr{H}(t - t_k)\,\mathbf{v}_j. \tag{549}$$

The inner convolution is easily evaluated using integration by parts, whose result is

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n}\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\int_{t_k}^{t}\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)$$
$$\Big[\hat{c}_{j,k,0}\exp\left(\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1}\exp\left(\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}^{-2}$$
$$\left. + \hat{c}_{j,k,1}t\exp\left(\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}^{-1}\Big]\right|_{t_k}^{t}\mathrm{d}t\,\mathscr{H}(t - t_k)\,\mathbf{v}_j, \tag{550}$$

that is equal to

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right) \int_{t_k}^{t} \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right) \exp\left(-\mathbf{F}_{2,1,1}t\right)$$
$$\Big[\hat{c}_{j,k,0} \exp\left(\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1} \exp\left(\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1} t \exp\left(\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}^{-1}$$
$$- \hat{c}_{j,k,0} \exp\left(\mathbf{F}_{2,1,1}t_k\right)\mathbf{F}_{2,1,1}^{-1} + \hat{c}_{j,k,1} \exp\left(\mathbf{F}_{2,1,1}t_k\right)\mathbf{F}_{2,1,1}^{-2}$$
$$- \hat{c}_{j,k,1} t_k \exp\left(\mathbf{F}_{2,1,1}t_k\right)\mathbf{F}_{2,1,1}^{-1}\Big] \mathrm{d}t \mathscr{H}\left(t - t_k\right)\mathbf{v}_j. \tag{551}$$

The remaining convolution can be split in two parts,

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right) \Bigg(\int_{t_k}^{t} \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right) \Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}$$
$$+ \hat{c}_{j,k,1} t \mathbf{F}_{2,1,1}^{-1}\Big] \mathrm{d}t - \int_{t_k}^{t} \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right) \exp\left(-\mathbf{F}_{2,1,1}t\right) \mathrm{d}t \exp\left(\mathbf{F}_{2,1,1}t_k\right) \Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}$$
$$- \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1} t_k \mathbf{F}_{2,1,1}^{-1}\Big] \mathrm{d}t\Bigg) \mathscr{H}\left(t - t_k\right)\mathbf{v}_j. \tag{552}$$

The first part can be calculated using integration by parts as for the inner convolution, while the second integral is the same as the one in Eq. (525), therefore, Equation (552) can be simplified to

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right) \Bigg(\Big[\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right) \Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} \Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}$$
$$- \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1} t \mathbf{F}_{2,1,1}^{-1}\Big] - \hat{c}_{j,k,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\Big)\Big]\Bigg|_{t_k}^{t}$$
$$- \left[\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t\right)\right]\Bigg|_{t_k}^{t} \exp\left(\mathbf{F}_{2,1,1}t_k\right) \Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}$$
$$+ \hat{c}_{j,k,1} t_k \mathbf{F}_{2,1,1}^{-1}\Big]\Bigg) \mathscr{H}\left(t - t_k\right)\mathbf{v}_j, \tag{553}$$

which is expanded to

$$\mathbf{y_p}(t)=\sum_{k=0}^{n_k}\sum_{j=1}^{n}\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\Big(\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} -\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t\mathbf{F}_{2,1,1}^{-1}\Big]-\hat{c}_{j,k,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\Big)-\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t_k\right)\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]-\hat{c}_{j,k,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\Big)- \Big[\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t\right)-\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t_k\right)\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}t_k\right)\Big] \exp\left(\mathbf{F}_{2,1,1}t_k\right)\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]\Big)\mathscr{H}\left(t-t_k\right)\mathbf{v}_j, \tag{554}$$

and, then, further simplified to

$$\mathbf{y_p}(t)=\sum_{k=0}^{n_k}\sum_{j=1}^{n}\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t\mathbf{F}_{2,1,1}^{-1}\Big] -\hat{c}_{j,k,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}-\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k-t)\right)\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} -\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]-\hat{c}_{j,k,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\Big)-\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k-t)\right) \Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k-t)\right)\mathbf{X}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} -\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]\Big)\mathscr{H}\left(t-t_k\right)\mathbf{v}_j, \tag{555}$$

that is rearranged again to

$$\mathbf{y_p}(t)=\sum_{k=0}^{n_k}\sum_{j=1}^{n}\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t\mathbf{F}_{2,1,1}^{-1}\Big] -\hat{c}_{j,k,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}-\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k-t)\right)\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} -\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]-\mathbf{X}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}+\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big] -\hat{c}_{j,k,1}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\Big)-\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k-t)\right)\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} +\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]\Big)\mathscr{H}\left(t-t_k\right)\mathbf{v}_j. \tag{556}$$

5.1.3.1 Solution due to first order HS when the modes are classical normal

Again, using the commutativity between $\mathbf{X}$ and $\mathbf{F}_{2,1,1}$, proven in Appendix D.1 when the system has classical normal modes, Equation (556) becomes

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n}\Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\left[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1}t\mathbf{F}_{2,1,1}^{-1}\right]$$
$$-\hat{c}_{j,k,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1} - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t)\right)\Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}$$
$$-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big] - \mathbf{X}\left[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\right]$$
$$-\hat{c}_{j,k,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\Big) - \exp\left(\mathbf{F}_{2,1,1}(t_k - t)\right)\mathbf{X}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2}$$
$$+\hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]\Big)\mathscr{H}(t - t_k)\,\mathbf{v}_j; \tag{557}$$

and substituting the solution **X** from Appendix D.1, Equation (557) expands to

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n}\Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\left[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1}t\mathbf{F}_{2,1,1}^{-1}\right]$$
$$-\hat{c}_{j,k,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1} - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t)\right)\Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}$$
$$-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big] - \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\left[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1} - \hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\right]$$
$$-\hat{c}_{j,k,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\Big) - \exp\left(\mathbf{F}_{2,1,1}(t_k - t)\right)\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\Big[\hat{c}_{j,k,0}\mathbf{F}_{2,1,1}^{-1}$$
$$-\hat{c}_{j,k,1}\mathbf{F}_{2,1,1}^{-2} + \hat{c}_{j,k,1}t_k\mathbf{F}_{2,1,1}^{-1}\Big]\Big)\mathscr{H}(t - t_k)\,\mathbf{v}_j, \tag{558}$$

that is, once more, equal to the results given in Chapter 3 and in Chapter 4, consequently, Equation (556) is consistent to classical normal modes too.

5.2 EFFICIENT COMPUTATION

Solving such systems of differential equations without classical normal modes poses new sources of computational cost, when compared to the results in Chapter 3 and in Chapter 4, namely the numerical solution of Eq. (514) and the solution of the Sylvester equation in Eq. (523). According to (DATTA, 2004; BARTELS; STEWART, 1972), the computation cost of solving the Sylvester equation is in the order of $\mathscr{O}\left(n^3\right)$ if the Schur decomposition is used to solve it. This is not particularly alarming, since this task is performed only once in a pre-processing step, as well as the numerical solution of Eq. (514). It was observed in Chapter 3 and in Chapter 4, through numerical experiments, that the computational cost tends to concentrate in the iterative processes of computing the response in the different time points, for the Generalized Integrating Factor and Heaviside Series as well as for numerical methods such as Newmark-beta.

Acknowledging the observation that the computation cost tends to lie mostly on the evaluation in different time points, for it grows as the number of points increases, there is a motivation to evaluate the response using the Generalized Integrating Factor or Heaviside Series in the most efficient way possible. As it was highlighted in Chapter 3 and in Chapter 4, one of the most efficient ways to tackle this problem is to evaluate the response at equally spaced points in time, so that the exponential map of the matrices is evaluated only once, as a pre-processing step. This procedure will be extended for the situation without classical normal modes in the following subsections.

The process of evaluating the exponential map of the matrix $\mathbf{A}$ in a time point t_{i+1}, forward by a distance of Δt of a already evaluated time point t_i is given by

$$\exp\left(\mathbf{A}t_{i+1}\right) = \exp\left(\mathbf{A}\left(\Delta t + t_i\right)\right) = \exp\left(\mathbf{A}\Delta t\right)\exp\left(\mathbf{A}t_i\right), \tag{559}$$

which implies, in a set of equally spaced time points, that the exponential map in a generic time point t_k equals

$$\exp\left(\mathbf{A}t_k\right) = \underbrace{\exp\left(\mathbf{A}\Delta t\right)\exp\left(\mathbf{A}\Delta t\right)\ldots\exp\left(\mathbf{A}\Delta t\right)}_{k\ times}\exp\left(\mathbf{A}t_0\right). \tag{560}$$

5.2.1 Homogeneous solution

Inspecting Eq. (526), there are two exponential maps, namely $\exp\left(-\mathbf{F}_{2,1,1}t\right)$ and $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)$. Each one can be separately evaluated using Eq. (560), just as with classical normal modes in Chapter 3 and in Chapter 4. The difference is in the matrix $\mathbf{X}$ at the left of $\exp\left(-\mathbf{F}_{2,1,1}t\right)$. As they do not commute, one would suggest that matrix-matrix multiplication would be necessary, but this is not the case, for the multiplication between $\exp\left(-\mathbf{F}_{2,1,1}t\right)$ and $\mathbf{C}_2$ can be carried out first and, then, the multiplication with $\mathbf{X}$. Hence, without classical normal modes, the number of matrix-vector multiplications grew from 2 to 3. The whole process is illustrated in Alg. 7. The 3 matrix-vector multiplications are highlighted in Alg. 7 and it is important to stress that the first and second matrix-vector multiplications are equally evaluated when the system does have classical normal modes.

Algorithm 4: Evaluation of the homogeneous response, $\mathbf{y_h}$, at equally spaced time points t_k

---

Calculate $\bar{\mathbf{K}}$, $\bar{\mathbf{C}}$, $\mathbf{F}_{2,1,1}$, $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)$, $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)$, $\mathbf{C}_2$ and $\mathbf{C}_1$ using Eq. (532)
Evaluate the homogeneous response at t_0
$\mathbf{y_h}\left(t_0\right) = \mathbf{X}\mathbf{C}_2 + \mathbf{C}_1$
for i=1,2,...,n_k
  Update the vectors $\mathbf{C}_1$ and $\mathbf{C}_2$
  $\mathbf{C}_1 = \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\mathbf{C}_1$, **1st matrix-vector multiplication**
  $\mathbf{C}_2 = \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\mathbf{C}_2$, **2nd matrix-vector multiplication**
  Calculate the homogeneous response
  $\mathbf{y_h}\left(t_i\right) = \mathbf{X}\mathbf{C}_2 + \mathbf{C}_1$, **3rd matrix-vector multiplication**
end

5.2.2 Particular solution due to Dirac's delta excitation

To simplify the response due to a series of Dirac's deltas, Eq. (542), the vector $\mathbf{v}_j$ can be placed inside the brackets, hence, it becomes

$$\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} c_{jk}\left[\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k - t)\right)\mathbf{v}_j - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t)\right)\mathbf{z}_j\right]\mathscr{H}(t - t_k), \tag{561}$$

where $\mathbf{z}_j = \mathbf{X}\mathbf{v}_j$. Now, the expression inside of the brackets resembles the mathematical structure of the homogeneous response, Eq. (526) and can be evaluated using the identity from Eq. (560). To perform this task, the most efficient way is to divide the intended simulation time into a set of equally spaced time points and the Dirac's deltas must be applied into the system in these time points. This approach enables the use of Eq. (561) to apply load and to measure the response. If a time point is intended just to assert the response of the system without any loading due to Dirac's delta, the coefficient c_{jk} has only to be zero.

Let Equation (561) be evaluated in a time point t_i. Due to the Heaviside step function properties, it equals

$$\mathbf{y_p}(t_i) = \sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\left[\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k - t_i)\right)\mathbf{v}_j - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_i)\right)\mathbf{z}_j\right]. \tag{562}$$

Let the term in brackets be split and rearranged, and let some terms be highlighted,

$$\mathbf{y_p}(t_i) = \mathbf{X}\underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\mathbf{F}_{2,1,1}(t_k - t_i)\right)\mathbf{v}_j}_{\boldsymbol{\gamma}_{1,i}} - \underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_i)\right)\mathbf{z}_j}_{\boldsymbol{\gamma}_{2,i}}. \tag{563}$$

Now, one shall evaluate Eq. (561) in a time point one step ahead, $t_{i+1} = t_i + \Delta t$,

$$\mathbf{y_p}(t_{i+1}) = \sum_{k=0}^{i+1}\sum_{j=1}^{n} c_{jk}\left[\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k - t_{i+1})\right)\mathbf{v}_j - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_{i+1})\right)\mathbf{z}_j\right], \tag{564}$$

This equation can have the term $i+1$ taken out of the summation,

$$\begin{aligned}\mathbf{y_p}(t_{i+1}) = &\sum_{j=1}^{n}\left[c_{j,i+1}\left[\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_{i+1} - t_{i+1})\right)\mathbf{v}_j - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_{i+1} - t_{i+1})\right)\mathbf{z}_j\right]\right] \\ &+ \sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\left[\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k - t_{i+1})\right)\mathbf{v}_j - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_{i+1})\right)\mathbf{z}_j\right].\end{aligned} \tag{565}$$

It follows from the definition of the vector $\mathbf{z}_j$ that the term taken out of the summation is indeed null; besides, the summation can be split in the same fashion of Eq. (563) and the exponential maps can, then, be factored,

$$\begin{aligned}\mathbf{y_p}\left(t_{i+1}\right) = \mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_i\right)\right)\mathbf{v}_j \\ - \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\left(t_k - t_i\right)\right)\mathbf{z}_j,\end{aligned} \tag{566}$$

which can be rewritten using the notation introduced in Eq. (563),

$$\mathbf{y_p}\left(t_{i+1}\right) = \mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i} - \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i}. \tag{567}$$

Equation (567) presents an interesting fact - the solution in a point ahead can be constructed just by scaling (through the exponential maps) information from an immediately previous point. Again, analogously to the homogeneous response, this evaluation is carried out by 3 matrix-vector multiplications.

Nevertheless, one must derive how the vectors $\boldsymbol{\gamma}_{1,i}$ and $\boldsymbol{\gamma}_{2,i}$ are updated from an iteration to the next. To this aim, let Equation (561) be evaluated in $t = t_{i+2} = t_i + 2\Delta t$,

$$\mathbf{y_p}\left(t_{i+2}\right) = \sum_{k=0}^{i+2}\sum_{j=1}^{n} c_{jk}\left[\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_{i+2}\right)\right)\mathbf{v}_j - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\left(t_k - t_{i+2}\right)\right)\mathbf{z}_j\right]; \tag{568}$$

again, Equation (568) can have the term $i+2$ taken out of the summation and the summation can be split and rearranged,

$$\begin{aligned}\mathbf{y_p}\left(t_{i+2}\right) = \sum_{j=1}^{n}\Big[c_{j,i+2}\left[\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}\left(t_{i+2} - t_{i+2}\right)\right)\mathbf{v}_j - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\left(t_{i+2} - t_{i+2}\right)\right)\mathbf{z}_j\right] \\ + \mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\sum_{k=0}^{i+1}\sum_{j=1}^{n} c_{jk}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_{i+1}\right)\right)\mathbf{v}_j \\ - \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\sum_{k=0}^{i+1}\sum_{j=1}^{n} c_{jk}\exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\left(t_k - t_{i+1}\right)\right)\mathbf{z}_j\Big].\end{aligned} \tag{569}$$

The term that multiplies $c_{j,i+2}$ is zero because of the definition of $\mathbf{z}_j$. Besides, if the the term $i+1$ is taken out of each one of the remaining summations, one gets

$$\mathbf{y_p}(t_{i+2}) = \mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\left[\sum_{j=1}^{n} c_{j,i+1}\exp\left(\mathbf{F}_{2,1,1}\left(t_{i+1}-t_{i+1}\right)\right)\mathbf{v}_j + \sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_{i+1}\right)\right)\mathbf{v}_j\right]$$
$$-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\left[\sum_{j=1}^{n} c_{j,i+1}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{i+1}-t_{i+1}\right)\right)\mathbf{z}_j + \sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k - t_{i+1}\right)\right)\mathbf{z}_j\right], \tag{570}$$

which is finally simplified to

$$\mathbf{y_p}(t_{i+2}) = \mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\left[\sum_{j=1}^{n} c_{j,i+1}\mathbf{v}_j + \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_i\right)\right)\mathbf{v}_j\right] - \exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)$$
$$\left[\sum_{j=1}^{n} c_{j,i+1}\mathbf{z}_j + \exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k - t_i\right)\right)\mathbf{z}_j\right]. \tag{571}$$

Comparing Eq. (571) to Eq. (567), one realizes that

$$\boldsymbol{\gamma}_{1,i+1} = \sum_{j=1}^{n} c_{j,i+1}\mathbf{v}_j + \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t_i\right)\right)\mathbf{v}_j,$$
$$\boldsymbol{\gamma}_{2,i+1} = \sum_{j=1}^{n} c_{j,i+1}\mathbf{z}_j + \exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\sum_{k=0}^{i}\sum_{j=1}^{n} c_{jk}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k - t_i\right)\right)\mathbf{z}_j; \tag{572}$$

which, by comparison with Eq. (563), yields the update rules for the vectors $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$,

$$\boldsymbol{\gamma}_{1,i+1} = \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i} + \sum_{j=1}^{n} c_{j,i+1}\mathbf{v}_j,$$
$$\boldsymbol{\gamma}_{2,i+1} = \exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i} + \sum_{j=1}^{n} c_{j,i+1}\mathbf{z}_j. \tag{573}$$

Inspecting Eq. (566) and Eq. (567), as well as Alg. 7, it is clear that these vectors must be initialized as

$$\boldsymbol{\gamma}_{1,0} = \sum_{j=1}^{n} c_{j,0}\mathbf{v}_j + \mathbf{C}_2,$$
$$\boldsymbol{\gamma}_{2,0} = \sum_{j=1}^{n} c_{j,0}\mathbf{z}_j - \mathbf{C}_1. \tag{574}$$

The whole process of evaluating Eq. (567) is illustrated in Alg. 5. There, the equations above are placed in a logic order, readily available for implementation. As it is common for a system not to be excited in all of its DOFs, a set of excited DOFs is created, where each excited DOF is numbered from 1 up to n_e, the size of this set.

Algorithm 5: Evaluation of the response $\mathbf{y_p}$ at the Dirac's delta singularity points t_k using Eq. (567)

Calculate $\bar{\mathbf{K}}$, $\bar{\mathbf{C}}$, $\mathbf{F}_{2,1,1}$, $\mathbf{X}$, $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)$ and $\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)$
Initialize a vector $n \times n_e 1$ for $\boldsymbol{\gamma}_1$ and for $\boldsymbol{\gamma}_2$
Iterate through the excited degrees of freedom n_e
for $j = 1, 2, \ldots, n_e$
    Computes $\mathbf{v}_j = \mathbf{M}^{-1}\mathbf{e}_j$ and $\mathbf{z}_j = \mathbf{X}\mathbf{v}_j$
    Initialize $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ using Eq. (574)
    Initialize the response as null and sums the initial conditions vectors $\mathbf{C}_1$ and $\mathbf{C}_2$
end
Iterate through the remaining points in time, $\{t_1, t_2, t_3, \ldots, t_{n_k}\}$
for $i = 1, 2, \ldots, n_k$
    Calculate $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_1$ and $\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_2$
    Calculate $\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_1$
    Calculate the response at time t_i, $\mathbf{y_p}(t_i)$, using Eq. (567)
    Iterate through the excited degrees of freedom
    Update $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ using Eq. (573)
end

5.2.3 Particular solution due to first order HS

In Chapter 4, the inverse operations were separated and simplified, the same can be done here. Thus, in Eq. (556), the definition of the vector $\mathbf{v}_j = \mathbf{M}^{-1}\mathbf{e}_j$ is applied and the matrix operations are expanded. Each one of these operations are named identically to that in Chapter 4,

$$
\begin{aligned}
\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n} \Bigg(& \hat{c}_{j,k,0} \underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_1} - \hat{c}_{j,k,1} \underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_3} \\
& + \hat{c}_{j,k,1} t \underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_1} - \hat{c}_{j,k,1} \underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_2} \\
& - \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k - t)\right) \Bigg(\hat{c}_{j,k,0} \underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_1} \\
& - \hat{c}_{j,k,1} \underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_3} + \hat{c}_{j,k,1} t_k \underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_1} - \hat{c}_{j,k,0} \underbrace{\mathbf{X}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_4} \\
& + \hat{c}_{j,k,1} \underbrace{\mathbf{X}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_5} - \hat{c}_{j,k,1} t_k \underbrace{\mathbf{X}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_4} \\
& - \hat{c}_{j,k,1} \underbrace{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_2} \Bigg) - \mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k - t)\right) \Bigg[\hat{c}_{j,k,0} \underbrace{\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_6} \\
& - \hat{c}_{j,k,1} \underbrace{\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_7} + \hat{c}_{j,k,1} t_k \underbrace{\mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1}}_{\boldsymbol{\Gamma}_6} \Bigg] \Bigg) \mathscr{H}(t - t_k)\,\mathbf{e}_j.
\end{aligned}
\tag{575}
$$

Using the property of inverse of matrices, namely $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, each term $\boldsymbol{\Gamma}_i$ can be rewritten or simplified. The simplifications for $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ were already developed in Chapter 4, resulting in

$$\boldsymbol{\Gamma}_1 = \mathbf{K}^{-1} = \boldsymbol{\Omega}_1^{-1}, \tag{576}$$

$$\boldsymbol{\Gamma}_2 = \left(\mathbf{K}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right)^{-1} = \boldsymbol{\Omega}_2^{-1}. \tag{577}$$

(578)

Let continue with $\boldsymbol{\Gamma}_3$,

$$\boldsymbol{\Gamma}_3 = \left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1} = \left(\mathbf{M}\mathbf{F}_{2,1,1}^{2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\right)^{-1} = \left(\mathbf{M}\mathbf{F}_{2,1,1}\bar{\mathbf{K}}\right)^{-1} = \boldsymbol{\Omega}_3^{-1}. \tag{579}$$

And with $\boldsymbol{\Gamma}_6$,

$$\boldsymbol{\Gamma}_6 = \mathbf{F}_{2,1,1}^{-1}\mathbf{M}^{-1} = \left(\mathbf{M}\mathbf{F}_{2,1,1}\right)^{-1} = \boldsymbol{\Omega}_6^{-1}. \tag{580}$$

Then, with $\mathbf{\Gamma}_7$,

$$\mathbf{\Gamma}_7 = \mathbf{F}_{2,1,1}^{-2}\mathbf{M}^{-1} = \left(\mathbf{M}\mathbf{F}_{2,1,1}^{2}\right)^{-1} = \mathbf{\Omega}_7^{-1}; \tag{581}$$

Finally, $\mathbf{\Gamma}_4$ and $\mathbf{\Gamma}_5$ are evaluated only by multiplying by $\mathbf{X}$, resulting in

$$\mathbf{\Gamma}_4 = \mathbf{X}\mathbf{\Omega}_6^{-1}, \tag{582}$$
$$\mathbf{\Gamma}_5 = \mathbf{X}\mathbf{\Omega}_7^{-1}. \tag{583}$$

Henceforth, alike in Chapter 4, all inverse operations can be carried out using a linear system of equations,

$$\mathbf{x}_{i,j} = \mathbf{\Omega}_i \backslash \mathbf{e}_j, \quad i = 1,2,3,4,5,6,7, \tag{584}$$

where $\backslash$ indicates the solution of a linear system $\mathbf{\Omega}_i\mathbf{x}_{i,j} = \mathbf{e}_j$, and $\mathbf{x}_{i,j}$ is the solution. Thus, substituting those results into Eq. (575), it yields

$$\begin{aligned}\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\sum_{j=1}^{n}\Big(\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t\right)\mathbf{x}_{1,j} - \hat{c}_{j,k,1}\mathbf{x}_{2,j} - \hat{c}_{j,k,1}\mathbf{x}_{3,j}\\ -\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k-t)\right)\Big(\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{1,j} - \hat{c}_{j,k,1}\mathbf{x}_{2,j} - \hat{c}_{j,k,1}\mathbf{x}_{3,j}\\ -\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j} + \hat{c}_{j,k,1}\mathbf{x}_{5,j}\Big)\\ -\mathbf{X}\exp\left(\mathbf{F}_{2,1,1}(t_k-t)\right)\left[\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j} - \hat{c}_{j,k,1}\mathbf{x}_{7,j}\right]\Big)\mathscr{H}(t-t_k).\end{aligned} \tag{585}$$

It follows directly from Eq. (582) and from Eq. (583) that

$$\begin{aligned}\mathbf{x}_{4,j} &= \mathbf{X}\mathbf{x}_{6,j},\\ \mathbf{x}_{5,j} &= \mathbf{X}\mathbf{x}_{7,j}.\end{aligned} \tag{586}$$

However, Eq. (587) can be regrouped to

$$\begin{aligned}\mathbf{y_p}(t) = \sum_{k=0}^{n_k}\left[t\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j} - \sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{8,j}\right]\mathscr{H}(t-t_k)\\ -\sum_{k=0}^{n_k}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](t_k-t)\right)\sum_{j=1}^{n}\Big(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}\\ +\hat{c}_{j,k,1}\mathbf{x}_{5,j} - \hat{c}_{j,k,1}\mathbf{x}_{8,j}\Big)\mathscr{H}(t-t_k) - \mathbf{X}\sum_{k=0}^{n_k}\exp\left(\mathbf{F}_{2,1,1}(t_k-t)\right)\sum_{j=1}^{n}\Big(\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j}\\ -\hat{c}_{j,k,1}\mathbf{x}_{7,j}\Big)\mathscr{H}(t-t_k),\end{aligned} \tag{587}$$

where

$$\mathbf{x}_{8,j} = \mathbf{x}_{2,j} + \mathbf{x}_{3,j}, \tag{588}$$

whose advantage is the requirement of just 6 vectors for each excited DOF, in opposition to the 7 vectors previously. This arrangement spares both the overall memory cost and the computational cost due to additions in each iteration.

Using the same technique deployed in the previous Subsec. 5.2.2 and in Chapter 4, let Eq. (587) be evaluated in a discrete set of equally spaced time points. Let it first be evaluated in a generic time point $t = t_i$ and have its terms regrouped,

$$\begin{aligned} \mathbf{y_p}(t_i) = \sum_{k=0}^{i}\left[t_i \sum_{j=1}^{n} \hat{c}_{j,k,1}\mathbf{x}_{1,j} + \sum_{j=1}^{n} \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \sum_{j=1}^{n} \hat{c}_{j,k,1}\mathbf{x}_{8,j}\right] \\ -\sum_{k=0}^{i} \exp\left(\left[\mathbf{C} - \bar{\mathbf{F}}_{\mathbf{2,1,1}}\right](t_k - t_i)\right) \sum_{j=1}^{n} \left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}\right. \\ \left. +\hat{c}_{j,k,1}\mathbf{x}_{5,j} - \hat{c}_{j,k,1}\mathbf{x}_{8,j}\right) - \mathbf{X}\sum_{k=0}^{i} \exp\left(\mathbf{F}_{2,1,1}(t_k - t_i)\right) \sum_{j=1}^{n} \left(\left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j} - \hat{c}_{j,k,1}\mathbf{x}_{7,j}\right) \end{aligned} \tag{589}$$

much like in Subsec. 5.2.2, two terms emerge

$$\begin{aligned} \boldsymbol{\gamma}_{2,i} = \sum_{k=0}^{i} \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_i)\right) \sum_{j=1}^{n} \left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}\right. \\ \left. +\hat{c}_{j,k,1}\mathbf{x}_{5,j} - \hat{c}_{j,k,1}\mathbf{x}_{8,j}\right), \\ \boldsymbol{\gamma}_{1,i} = \sum_{k=0}^{i} \exp\left(\mathbf{F}_{2,1,1}(t_k - t_i)\right) \sum_{j=1}^{n} \left(\left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j} - \hat{c}_{j,k,1}\mathbf{x}_{7,j}\right). \end{aligned} \tag{590}$$

Now, let Eq. (587) be evaluated in $t = t_{i+1} = t_i + \Delta t$,

$$\begin{aligned} \mathbf{y_p}(t_{i+1}) = \sum_{k=0}^{i+1}\left[t_{i+1} \sum_{j=1}^{n} \hat{c}_{j,k,1}\mathbf{x}_{1,j} + \sum_{j=1}^{n} \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \sum_{j=1}^{n} \hat{c}_{j,k,1}\mathbf{x}_{8,j}\right] \\ -\sum_{k=0}^{i+1} \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right](t_k - t_{i+1})\right) \sum_{j=1}^{n} \left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \hat{c}_{j,k,0}\mathbf{x}_{1,j} - \left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}\right. \\ \left. +\hat{c}_{j,k,1}\mathbf{x}_{5,j} - \hat{c}_{j,k,1}\mathbf{x}_{8,j}\right) - \mathbf{X}\sum_{k=0}^{i+1} \exp\left(\mathbf{F}_{2,1,1}(t_k - t_{i+1})\right) \sum_{j=1}^{n} \left(\left(\hat{c}_{j,k,0} + \hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j}\right. \\ \left. -\hat{c}_{j,k,1}\mathbf{x}_{7,j}\right), \end{aligned} \tag{591}$$

whose term with $k=i+1$ can be taken out of the summation, while the summation is split in the same fashion as in Eq. (589),

$$\begin{aligned}
\mathbf{y_p}\left(t_{i+1}\right) = t_{i+1}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j} + \sum_{j=1}^{n}\hat{c}_{j,i+1,0}\mathbf{x}_{1,j} - \sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{8,j} - t_{i+1}\sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{1,j}\\
-\sum_{j=1}^{n}\hat{c}_{j,i+1,0}\mathbf{x}_{1,j} + \sum_{j=1}^{n}\left(\hat{c}_{j,i+1,0}+\hat{c}_{j,i+1,1}t_{i+1}\right)\mathbf{x}_{4,j} - \sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{5,j} + \sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{8,j}\\
-\sum_{j=1}^{n}\left(\hat{c}_{j,i+1,0}+\hat{c}_{j,i+1,1}t_{i+1}\right)\mathbf{X}\mathbf{x}_{6,j} + \sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{X}\mathbf{x}_{7,j} + \sum_{k=0}^{i}\left[t_{i+1}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}\right.\\
\left.+\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j} - \sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{8,j}\right] - \sum_{k=0}^{i}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t_{i+1}\right)\right)\sum_{j=1}^{n}\left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}\right.\\
\left.+\hat{c}_{j,k,0}\mathbf{x}_{1,j} - \left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j} + \hat{c}_{j,k,1}\mathbf{x}_{5,j} - \hat{c}_{j,k,1}\mathbf{x}_{8,j}\right)\\
-\mathbf{X}\sum_{k=0}^{i}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_{i+1}\right)\right)\sum_{j=1}^{n}\left(\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j} - \hat{c}_{j,k,1}\mathbf{x}_{7,j}\right),
\end{aligned} \tag{592}$$

the elements out of the summation in index k cancel each other out using relations of Eq. (586) and of Eq. (588). Equation (592) can, then, be further simplified to

$$\begin{aligned}
\mathbf{y_p}\left(t_{i+1}\right) = t_{i+1}\underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}}_{\mathbf{w}_{1,i}} + \underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}}_{\mathbf{w}_{2,i}} - \underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{8,j}}_{\mathbf{w}_{3,i}}\\
-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\underbrace{\left\{\begin{matrix}\sum_{k=0}^{i}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t_i\right)\right)\sum_{j=1}^{n}\left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\hat{c}_{j,k,0}\mathbf{x}_{1,j}\right.\\ \left.-\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\mathbf{x}_{5,j}-\hat{c}_{j,k,1}\mathbf{x}_{8,j}\right)\end{matrix}\right\}}_{\boldsymbol{\gamma}_{2,i}}\\
-\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\underbrace{\sum_{k=0}^{i}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_i\right)\right)\sum_{j=1}^{n}\left(\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j}-\hat{c}_{j,k,1}\mathbf{x}_{7,j}\right)}_{\boldsymbol{\gamma}_{1,i}}
\end{aligned} \tag{593}$$

As it was made in Subsec. 5.2.2, Equation (593) can be rewritten using relations from Eq. (590),

$$\mathbf{y_p}\left(t_{i+1}\right) = t_{i+1}\mathbf{w}_{1,i} + \mathbf{w}_{2,i} - \mathbf{w}_{3,i} - \exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i} - \mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}. \tag{594}$$

In Subsec. 5.2.2, the response was also evaluated in time $t=t_{i+2}$ to derive how the vectors $\boldsymbol{\gamma}_{2,i}$ and $\boldsymbol{\gamma}_{1,i}$ are updated. The same intuition will be used ahead, hence,

$$\mathbf{y_p}\left(t_{i+2}\right)=t_{i+2}\sum_{k=0}^{i+2}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\sum_{k=0}^{i+2}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}-\sum_{k=0}^{i+2}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{8,j}$$
$$-\sum_{k=0}^{i+2}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t_{i+2}\right)\right)\sum_{j=1}^{n}\left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\hat{c}_{j,k,0}\mathbf{x}_{1,j}-\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}\right.$$
$$\left.+\hat{c}_{j,k,1}\mathbf{x}_{5,j}-\hat{c}_{j,k,1}\mathbf{x}_{8,j}\right)-\mathbf{X}\sum_{k=0}^{i+2}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_{i+2}\right)\right)\sum_{j=1}^{n}\left(\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j}\right.$$
$$\left.-\hat{c}_{j,k,1}\mathbf{x}_{7,j}\right). \tag{595}$$

Again, taking the term $i+2$ out of the summation in k and separating the summation in three components,

$$\mathbf{y_p}\left(t_{i+2}\right)=t_{i+2}\sum_{j=1}^{n}\hat{c}_{j,i+2,1}\mathbf{x}_{1,j}+\sum_{j=1}^{n}\hat{c}_{j,i+2,0}\mathbf{x}_{1,j}-\sum_{j=1}^{n}\hat{c}_{j,i+2,1}\mathbf{x}_{8,j}-t_{i+2}\sum_{j=1}^{n}\hat{c}_{j,i+2,1}\mathbf{x}_{1,j}$$
$$-\sum_{j=1}^{n}\hat{c}_{j,i+2,0}\mathbf{x}_{1,j}+\sum_{j=1}^{n}\left(\hat{c}_{j,i+2,0}+\hat{c}_{j,i+2,1}t_{i+2}\right)\mathbf{x}_{4,j}-\sum_{j=1}^{n}\hat{c}_{j,i+2,1}\mathbf{x}_{5,j}+\sum_{j=1}^{n}\hat{c}_{j,i+2,1}\mathbf{x}_{8,j}$$
$$-\sum_{j=1}^{n}\left(\hat{c}_{j,i+2,0}+\hat{c}_{j,i+2,1}t_{i+2}\right)\mathbf{X}\mathbf{x}_{6,j}+\sum_{j=1}^{n}\hat{c}_{j,i+2,1}\mathbf{X}\mathbf{x}_{7,j}+t_{i+2}\sum_{k=0}^{i+1}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}$$
$$+\sum_{k=0}^{i+1}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}-\sum_{k=0}^{i+1}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{8,j}-\sum_{k=0}^{i+1}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t_{i+2}\right)\right)\sum_{j=1}^{n}\left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}\right.$$
$$\left.+\hat{c}_{j,k,0}\mathbf{x}_{1,j}-\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\mathbf{x}_{5,j}-\hat{c}_{j,k,1}\mathbf{x}_{8,j}\right)$$
$$-\mathbf{X}\sum_{k=0}^{i+1}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_{i+2}\right)\right)\sum_{j=1}^{n}\left(\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j}-\hat{c}_{j,k,1}\mathbf{x}_{7,j}\right). \tag{596}$$

Once more, the terms out of the summation cancel each other out using the relations of Eq. (586) and of Eq. (588). Thus, Equation (596) can be simplified and the exponential maps can be rearranged to

$$\mathbf{y_p}\left(t_{i+2}\right)=t_{i+2}\sum_{k=0}^{i+1}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\sum_{k=0}^{i+1}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}-\sum_{k=0}^{i+1}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{8,j}$$
$$-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\sum_{k=0}^{i+1}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t_{i+1}\right)\right)\sum_{j=1}^{n}\left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\hat{c}_{j,k,0}\mathbf{x}_{1,j}\right.$$
$$\left.-\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\mathbf{x}_{5,j}-\hat{c}_{j,k,1}\mathbf{x}_{8,j}\right)$$
$$-\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\sum_{k=0}^{i+1}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_{i+1}\right)\right)\sum_{j=1}^{n}\left(\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j}-\hat{c}_{j,k,1}\mathbf{x}_{7,j}\right); \tag{597}$$

taking the terms $i+1$ out of the summations and rearranging the exponential maps again, one gets

$$\mathbf{y_p}\left(t_{i+2}\right) = t_{i+2}\underbrace{\left(\sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{1,j} + \underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{1,j}}_{\mathbf{w}_{1,i}}\right)}_{\mathbf{w}_{1,i+1}}$$
$$+\underbrace{\left(\sum_{j=1}^{n}\hat{c}_{j,i+1,0}\mathbf{x}_{1,j} + \underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,0}\mathbf{x}_{1,j}}_{\mathbf{w}_{2,i}}\right)}_{\mathbf{w}_{2,i+1}} - \underbrace{\left(\sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{8,j} + \underbrace{\sum_{k=0}^{i}\sum_{j=1}^{n}\hat{c}_{j,k,1}\mathbf{x}_{8,j}}_{\mathbf{w}_{3,i}}\right)}_{\mathbf{w}_{3,i+1}}$$
$$-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\left[t_{i+1}\sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{1,j} + \sum_{j=1}^{n}\hat{c}_{j,i+1,0}\mathbf{x}_{1,j}\right.$$
$$-\sum_{j=1}^{n}\left(\hat{c}_{j,i+1,0}+\hat{c}_{j,i+1,1}t_{i+1}\right)\mathbf{x}_{4,j} + \sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{5,j} - \sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{8,j}$$
$$+\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\sum_{k=0}^{i}\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_k-t_i\right)\right)\sum_{j=1}^{n}\left(t_k\hat{c}_{j,k,1}\mathbf{x}_{1,j}+\hat{c}_{j,k,0}\mathbf{x}_{1,j}-\right.$$
$$\left.\left.\left(\hat{c}_{j,k,0}+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{4,j}+\hat{c}_{j,k,1}\mathbf{x}_{5,j}-\hat{c}_{j,k,1}\mathbf{x}_{8,j}\right)\right] - \mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\left[\sum_{j=1}^{n}\left(\hat{c}_{j,i+1,0}\right.\right.$$
$$\left.+\hat{c}_{j,i+1,1}t_{i+1}\right)\mathbf{x}_{6,j} - \sum_{j=1}^{n}\hat{c}_{j,i+1,1}\mathbf{x}_{7,j} + \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\sum_{k=0}^{i}\exp\left(\mathbf{F}_{2,1,1}\left(t_k-t_i\right)\right)\sum_{j=1}^{n}\left(\left(\hat{c}_{j,k,0}\right.\right.$$
$$\left.\left.\left.+\hat{c}_{j,k,1}t_k\right)\mathbf{x}_{6,j}-\hat{c}_{j,k,1}\mathbf{x}_{7,j}\right)\right], \tag{598}$$

which is similar to Eq. (593) and, consequently, can be rewritten to

$$\begin{aligned}\mathbf{y_p}\left(t_{i+2}\right) &= t_{i+2}\mathbf{w}_{1,i+1} + \mathbf{w}_{2,i+1} - \mathbf{w}_{3,i+1}\\ &-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i+1} - \mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i+1}.\end{aligned} \tag{599}$$

Finally, Eq. (599) leads to the update rules for the vectors $\boldsymbol{\gamma}_{2,i}$, $\boldsymbol{\gamma}_{1,i}$, $\mathbf{w}_{1,i}$, $\mathbf{w}_{2,i}$ and $\mathbf{w}_{3,i}$,

$$
\begin{aligned}
\mathbf{w}_{1,i+1} &= \mathbf{w}_{1,i} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1}\mathbf{x}_{1,j}}_{\mathbf{d}_{1,i+1}}, \\
\mathbf{w}_{2,i+1} &= \mathbf{w}_{2,i} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,0}\mathbf{x}_{1,j}}_{\mathbf{d}_{2,i+1}}, \\
\mathbf{w}_{3,i+1} &= \mathbf{w}_{3,i} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1}\mathbf{x}_{8,j}}_{\mathbf{d}_{3,i+1}}, \\
\boldsymbol{\gamma}_{2,i+1} &= \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i} + t_{i+1}\underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1}\mathbf{x}_{1,j}}_{\mathbf{d}_{1,i+1}} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,0}\mathbf{x}_{1,j}}_{\mathbf{d}_{2,i+1}} \\
&\quad - \underbrace{\sum_{j=1}^{n}\left(\hat{c}_{j,i+1,0} + \hat{c}_{j,i+1,1}t_{i+1}\right)\mathbf{x}_{4,j}}_{\mathbf{d}_{4,i+1}} + \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1}\mathbf{x}_{5,j}}_{\mathbf{d}_{5,i+1}} - \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1}\mathbf{x}_{8,j}}_{\mathbf{d}_{3,i+1}}, \\
\boldsymbol{\gamma}_{1,i+1} &= \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i} + \underbrace{\sum_{j=1}^{n}\left(\hat{c}_{j,i+1,0} + \hat{c}_{j,i+1,1}t_{i+1}\right)\mathbf{x}_{6,j}}_{\mathbf{d}_{6,i+1}} - \underbrace{\sum_{j=1}^{n} \hat{c}_{j,i+1,1}\mathbf{x}_{7,j}}_{\mathbf{d}_{7,i+1}};
\end{aligned}
\tag{600}
$$

which is then rewritten as

$$
\begin{aligned}
\mathbf{w}_{1,i+1} &= \mathbf{w}_{1,i} + \mathbf{d}_{1,i+1}, \\
\mathbf{w}_{2,i+1} &= \mathbf{w}_{2,i} + \mathbf{d}_{2,i+1}, \\
\mathbf{w}_{3,i+1} &= \mathbf{w}_{3,i} + \mathbf{d}_{3,i+1}, \\
\boldsymbol{\gamma}_{2,i+1} &= \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i} + t_{i+1}\mathbf{d}_{1,i+1} + \mathbf{d}_{2,i+1} - \mathbf{d}_{4,i+1} + \mathbf{d}_{5,i+1} - \mathbf{d}_{3,i+1}, \\
\boldsymbol{\gamma}_{1,i+1} &= \exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i} + \mathbf{d}_{6,i+1} - \mathbf{d}_{7,i+1}.
\end{aligned}
\tag{601}
$$

The structure of the vectors $\mathbf{d}$ and their construction in matrix form was already discussed in Chapter 4. Inspecting Eq. (601), one observes that, when the modes are not classical normal, there are 7 matrix-vector multiplications ($n \times n_e \cdot n \times 1$, where n_e is the number of excited DOFs) and 3 matrix-vector multiplications ($n \times n \cdot n \times 1$) per iteration. The quantity n_e was introduced in Chapter 4 too as the cardinality of the set of excited DOFs, $\mathscr{S}$. As $n_e \leq n$, the complexity of the procedure is dominated by the 3 matrix-vector multiplications ($n \times n \cdot n \times 1$), namely $\exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_{2,i}$, $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}$ and $\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}$, as their complexity is $\mathscr{O}\left(n^2\right)$. Hence, when the system does not have classical normal modes, the cost of the Heaviside series method is 50% higher compared to when it does.

The third multiplication $\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}$ can be evaluated in a more rational way. As $\mathbf{X}$ is invariant to the iteration, the vectors $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}$ could be stored and the multiplication could, then, be carried out later as a post-processing step,

$$\begin{aligned}&\left[\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,0}\;\;\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,1}\;\;\ldots\;\;\mathbf{X}\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,n_k-1}\right]=\\&\quad\mathbf{X}\underbrace{\left[\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,0}\;\;\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,1}\;\;\ldots\;\;\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,n_k-1}\right]}_{\mathbf{E}}.\end{aligned}\tag{602}$$

Multiplying $\mathbf{X}$ by the vector $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_{1,i}$ at each iteration is essentially the naive method to multiply the matrix $\mathbf{X}$ by the matrix $n\times n_k$ $\mathbf{E}$. However, there are other numerical methods to do that multiplication much more efficiently and, consequently, doing it in that way, the additional computational effort compared to Heaviside series with classical normal modes is less than 50% (the exact figure relying on the matrix-matrix multiplication method).

In Chapter 3 and in Chapter 4, it was stated that the initial conditions of the particular response due to HS are always homogeneous, this reasoning can be used again, for non-classical normal modes, since the nature of the vibration modes were not prescribed for deriving the initial conditions in Chapter 4. One might observe that the mathematical structure of the homogeneous solution in Alg. 7 is very similar to Eq. (594), therefore, the vectors of integration constants, $\mathbf{C}_1$ and $\mathbf{C}_2$, can be incorporated directly into the initialization of the vectors $\boldsymbol{\gamma}_{1,i}$ and $\boldsymbol{\gamma}_{2,i}$. Finally, it is clear that the parameters in Eq. (601) must be initialized as follows, $t_0 = 0$,

$$\begin{aligned}\mathbf{w}_{1,0} &= \mathbf{d}_{1,0},\\ \mathbf{w}_{2,0} &= \mathbf{d}_{2,0},\\ \mathbf{w}_{3,0} &= \mathbf{d}_{3,0},\\ \boldsymbol{\gamma}_{2,0} &= \mathbf{d}_{2,0}-\mathbf{d}_{4,0}+\mathbf{d}_{5,0}-\mathbf{d}_{3,0}-\mathbf{C}_1,\\ \boldsymbol{\gamma}_{1,0} &= \mathbf{d}_{6,0}-\mathbf{d}_{7,0}-\mathbf{C}_2.\end{aligned}\tag{603}$$

With the addition of the integration constant vectors to the vectors $\boldsymbol{\gamma}_{1,0}$ and $\boldsymbol{\gamma}_{2,0}$ in Eq. (603), the homogeneous response can be evaluated in the same steps where the particular solution is calculated, thus, no extra operations are needed and the computational cost decreases.

Due to the property of the particular solution using HS being homogeneous at $t = t_0$, the evaluation of the integrating constants in Eq. (532) is even simpler,

$$\begin{aligned}\mathbf{C}_2 &= \mathbf{v}+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\mathbf{u},\\ \mathbf{C}_1 &= \mathbf{u}-\mathbf{X}\mathbf{C}_2.\end{aligned}\tag{604}$$

Algorithm 6: Evaluation of the response $\mathbf{y_p}$ at the Heaviside singularity points t_k using Eq. (594) and the update rules from Eq. (601)

Calculate $\bar{\mathbf{K}}$, $\bar{\mathbf{C}}$, $\mathbf{F}_{2,1,1}$, $\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)$ and $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)$
Calculate the vectors of initial conditions, $\mathbf{C}_1$ and $\mathbf{C}_2$, using Eq. (604)
Initialize a vector $n\times 1$ for each vector $\mathbf{d}$
Initialize a vector $n\times 1$ for each vector $\mathbf{w}$
Initialize a vector $n\times 1$ for $\boldsymbol{\gamma}_1$ and for $\boldsymbol{\gamma}_2$
Initialize a matrix $n\times n_e$ for each of $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_4$ and $\mathbf{X}_5$
Calculate $\boldsymbol{\Omega}_1$, $\boldsymbol{\Omega}_2$, $\boldsymbol{\Omega}_3$, $\boldsymbol{\Omega}_4$, $\boldsymbol{\Omega}_5$, $\boldsymbol{\Omega}_6$ and $\boldsymbol{\Omega}_7$
Iterate through the set of excited degrees of freedom $\mathscr{S}$
for $j=1,2,\ldots,n_e$
    Calculate $\mathbf{X}_i\left[j\right]=\boldsymbol{\Omega}_i\backslash\mathbf{e}_j$
    Calculate coefficients $\hat{c}_{j,0,0}$ and $\hat{c}_{j,0,1}$ as in Chapter 4
    Initialize $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ using Eq. (603)
    Initialize the response as null and sums the initial conditions vectors, $\mathbf{C}_1$ and $\mathbf{X}\mathbf{C}_2$
end
Iterate through the remaining points in time, $\{t_1,t_2,t_3,\ldots,t_{n_k}\}$
for $i=1,2,\ldots,n_k$
    Calculate $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\boldsymbol{\gamma}_1$ and $\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\boldsymbol{\gamma}_2$
    Calculate the response at time t_i, $\mathbf{y_p}\left(t_i\right)$, using Eq. (594)
    Iterate through the set of excited degrees of freedom $\mathscr{S}$
    for $j=1,2,\ldots,n_e$
        Calculate coefficients $\hat{c}_{ji0}$ and $\hat{c}_{ji1}$ as in Chapter 4
    end
    Update $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, $\mathbf{d}_4$, $\mathbf{d}_5$, $\mathbf{d}_6$, $\mathbf{d}_7$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ using Eq. (601)
end

5.3 STABILITY OF THE SOLUTION DUE TO FIRST ORDER HS

Inspecting Eq. (526) and Eq. (594), one observes that both the homogeneous solution and the particular solution due to first order HS share the same mathematical structure, *i.e.* the multiplication of two exponential maps, $\exp\left(-\mathbf{A}t\right)$, by their respective vectors. It was proven in Chapter 4 that, if the eigenvalues of $\mathbf{A}$ have positive real part, the solution is unconditionally stable, consequently, no matter how large is the time step, Δt, the solution is bounded. The same applies here, for systems without classical normal modes, since the only difference is the multiplication of the solvent of the Sylvester equation, $\mathbf{X}$, in front of the term with $\exp\left(-\mathbf{F}_{2,1,1}t\right)$, which is a constant matrix throughout the time domain.

It is important to stress that, as demonstrated in Chapter 4, the eigenvalues of $\mathbf{F}_{2,1,1}$ and of $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$ are not a shortcoming of the method, either GIF nor HS, since those eigenvalues are characteristic to the system to be solved and its modelling. For instance, as shown in Chapter 4 and in Appendix C.3, the Rayleigh damping can produce eigenvalues with negative real part, which is, then, a form of *negative damping*, since it gives energy to the system instead of dissipating it away.

5.4 FINAL REMARKS OF THE CHAPTER

The solution to second order systems of linear ODEs with constant matrix coefficients was tackled when such systems do not present classical normal modes. It was shown that the solutions are consistent to the approach delivered in the previous chapters and that those solutions are indeed particular cases of the newly presented ones. The main difference from the solutions of this chapter to the previous ones is the accessory Sylvester equation, whose solution procedure was discussed and it was stressed that it must be carried out only once in a preprocessing step.

Computational aspects of the solutions were discussed and it was shown that the complexity analysis has not increased in order, which is especially important since the main source of computational cost uses to be the iterative process along the time points. It was also demonstrated that the multiplication of $\mathbf{X}$ by the exponential map can be carried out in a post-processing step using State of the Art algorithms for multiplication between rectangular matrices, instead of naively multiplying $\mathbf{X}$ in every single iteration.

Numerical experiments as the ones documented in Chapter 3 and in Chapter 4 are left for future works. The future experiments will compare different approaches to solve the quadratic matrix equation numerically too. Then, a picture of the total computational cost will be available.

6 CONCLUSION

This work proposes a new family of methods to solve both one-dimensional and systems of coupled linear ODEs of order $n \geq 2$. This family is constructed upon a generalization of the Leibniz integrating factor for first order linear differential equations. The method is a systematic approach for order reduction that can systematically find analytical complete solutions of linear ordinary differential equations with or without constant coefficients.

The methodology was applied to second-order ordinary differential equation, since this type of ODE is of great importance in Applied Mathematics, Physics and Engineering. Special care was devoted to the constant coefficient case for different types of excitations. Although, successful examples were provided for ODEs with coefficients that are indeed functions of the independent variable.

The solution procedure requires the particular solution of a sister nonlinear ODE, which is a Riccati ODE in the case of second order ODEs, for one-dimensional and for systems of coupled ODEs. It was shown that, for many important ODEs in Engineering, the particular solution to the Riccati equation is easily found; as is the situation with constant coefficients, when the nonlinear sister equation transforms into a simple quadratic algebraic equation.

The GIF method was proved to give the homogeneous and particular solutions independently using double convolutions for second order ODEs. These double convolutions were particularized for important cases of excitation functions in Engineering, such as continuous functions - like periodic, exponential, and polynomial - and as discontinuous functions - like Dirac's delta and Heaviside step.

In the case of systems of coupled ODEs with constant coefficients, the homogeneous solution and the particular solution due to discontinuous excitation functions were shown to be made out of exponential maps. Due to the inherent computational cost of evaluating exponential maps, an optimized way to calculate them in a discrete set of points was proposed to greatly reduce computational effort. Thereby, important real-world problems can be analytically solved by using the proposed approach.

The computational complexity of the proposed approach was compared to other analytical and numerical methods. It was shown that the Laplace transform is not suitable to analytically solve large problems due to the symbolic inverses. The complexity associated to matrix exponentials, matrix multiplications and inverses also makes the State Variables approach too costly when the dimensionality of the problem increases. Furthermore, a computational experiment was carried out to compare evaluation time of the proposed approach to the State Variables and to the Newmark-beta methods. It was shown that the proposed method reduced the evaluation time considerably, thus, being a reliable and viable method for the analytical solution and computational simulation of large systems.

Despite these advantages, sometimes it is not feasible to evaluate the double convolutions to get the particular solutions or it is simply not desired to. In such cases, the GIF must be extended to cope with and, to this aim, the HS method was developed. The HS approach approximates the original excitation function using a finite series of Heaviside steps multiplied by polynomial terms, which has analytical solution using the GIF. Thus, the problem of solving the double convolutions is substituted by a problem of approximating the original excitation, that is done fairly easily.

It was demonstrated, both from algorithmic complexity analysis and through numeric experiments, that the HS method kept the computational efficiency of the GIF. The HS also presented astounding accuracy, with rates of convergence between 2 and 4 against the rates of 1 and 2 of Newmark-beta method. It was also proven that the HS is unconditionally stable if the system is physically stable, *i.e.* the HS and the GIF methods do not have artificial numerical damping and, consequently, do not hide ill-conditioned modelling.

Results of Chapter 3 and of Chapter 4 were derived using the hypothesis of classical normal modes, which is widely used in most of vibration analysis. However, to demonstrate that the proposed family of methods is not limited to this hypothesis, Chapter 5 was issued to investigate the behavior and feasibility of the solutions when the system does not have classical normal modes. It was shown that the solution procedure depends upon a Sylvester equation that must be tackled only once. Besides, the computational cost per iteration is not increased much, while further implementation suggestions were made.

As final conclusion, one can say that the proposed family of methods is widely applicable to solve real-world problems, most importantly, with greater accuracy and lower computational cost. Future works are plenty: do numerical experiments with GIF and HS when the system does not have classical normal modes, extend the HS method to nonlinear regime, investigate applications in modal analysis and other computationally intensive problems, such as Structural Optimization and BEA.

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APPENDIX A – AUXILIARY FORMULATION FOR THE GENERALIZED INTEGRATING FACTOR FOR LINEAR ODES WITH A SINGLE DOF

A.1 FAMILIES OF RICCATI EQUATIONS WITH SIMPLE PARTICULAR SOLUTIONS USING SUBSTITUTION

Although the Riccati sister equation, Eq. (34), is a nonlinear differential equation, for many cases finding a particular solution is straightforward. Two cases will be addressed: when the particular solution is a constant and when the particular solution is a polynomial of the independent variable. The focus will be on Eq. (34), nonetheless, the analysis holds true for the equation with $f_{2,1,1}$.

A.1.1 Constant particular solution

To Eq. (34) admit a constant particular solution, the following conditions must hold:

$$c - \dot{m} = a, \tag{605}$$

and,

$$km = b, \tag{606}$$

where a and b are constants. Thus, Equation (34) is simplified to

$$f_{2,1,2}^2 = a f_{2,1,2} - b \tag{607}$$

whose particular solutions are the roots of the above quadratic algebraic equation. The positive root is chosen,

$$f_{2,1,2} = \frac{a + \sqrt{a^2 - 4b}}{2}. \tag{608}$$

A.1.2 Polynomial particular solution

Let $f_{2,1,2}$ and the differential equation coefficients be polynomials,

$$f_{2,1,2}(t) = z_0 + z_1 t + z_2 t^2 + \cdots + z_a t^a, \tag{609}$$

$$m(t) = m_0 + m_1 t + m_2 t^2 + \cdots + m_b t^b, \tag{610}$$

$$c(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_d t^d, \tag{611}$$

$$k(t) = k_0 + k_1 t + k_2 t^2 + \cdots + k_g t^g. \tag{612}$$

Consequentely the polynomial degree, $D(p(t))$, of each of these coefficients and particular solution is: $D(f_{2,1,2}) = a$, $D(m) = b$, $D(c) = d$ and $D(k) = g$. When applying these polynomials to Eq. (34), the degrees of the polynomials to be equal between the left hand side and the right hand side, the degrees must satisfy the following relations:

$$a = b-1; \;\; d \leq b-1; \;\; g \leq b-2, \;\; \text{or} \tag{613}$$
$$a = d; \;\; b \leq d-1; \;\; g \leq 2d-b, \;\; \text{or} \tag{614}$$
$$a = \frac{b+g}{2}; \;\; b \leq g+2; \;\; d \leq \frac{b+g}{2}. \tag{615}$$

Thus, applying these polynomials into Eq. (34) and separating the squares of each coefficient of the solution yields

$$\begin{aligned} z_0^2 + z_1^2 t^2 + z_2^2 t^4 + \cdots + z_a^2 t^{2a} - z_0^2 - z_1^2 t^2 - z_2^2 t^4 - \cdots - z_a^2 t^{2a} + f_{2,1,2}^2 = \\ \left(m_0 + m_1 t + m_2 t^2 + \cdots + m_b t^b\right)\left(z_1 + 2z_2 t + 3z_3 t^2 + \cdots + a z_a t^{a-1}\right) + \\ \left(c_0 - m_1 + t\left(c_1 + 2m_2\right) + t^2\left(c_2 + 3m_3\right) + \cdots\right)\left(z_0 + z_1 t + z_2 t^2 + \cdots + z_a t^a\right) - \\ -m_0 k_0 - t\left(m_0 k_1 + m_1 k_0\right) - t^2\left(m_0 k_2 + m_1 k_1 + m_2 k_0\right) - \cdots . \end{aligned} \tag{616}$$

The even powers of t on the left hand side can be matched to the equivalent terms on the right hand side,

$$\begin{aligned} z_0^2 + z_1^2 t^2 + z_2^2 t^4 + \cdots + z_a^2 t^{2a} - z_0^2 - z_1^2 t^2 - z_2^2 t^4 - \cdots - z_a^2 t^{2a} + f_{2,1,2}^2 = \\ m_2 z_1 t^2 + 2m_3 z_2 t^4 + \cdots + a m_a z_b t^{a+b-1} - m_2 z_1 t^2 - 2m_3 z_2 t^4 - \cdots - a m_a z_b t^{a+b-1} + \\ m\dot{f}_{2,1,2} + z_0\left(c_0 - m_1\right) + z_1 t^2\left(c_1 - 2m_2\right) + z_2 t^4\left(c_2 - 3m_3\right) + \cdots - z_0\left(c_0 - m_1\right) \\ -z_1 t^2\left(c_1 - 2m_2\right) - z_2 t^4\left(c_2 - 3m_3\right) - \cdots + \left(c - \dot{m}\right) f_{2,1,2} - m_0 d_0 - \\ t^2\left(m_0 d_2 + m_1 d_1 + m_2 d_0\right) - t^4\left(m_0 d_4 + m_1 d_3 + m_2 d_2 + m_3 d_1 + m_4 d_0\right) - \cdots + \\ m_0 d_0 + t^2\left(m_0 d_2 + m_1 d_1 + m_2 d_0\right) + t^4\left(m_0 d_4 + m_1 d_3 + m_2 d_2 + m_3 d_1 + m_4 d_0\right) + \\ \cdots - md. \end{aligned} \tag{617}$$

where from, one can derive the following independent relations,

$$\begin{aligned} z_0^2 &= z_0\left(c_0 - m_1\right) - m_0 d_0, \\ z_1^2 &= z_1\left(c_1 - m_2\right) - m_0 d_2 - m_1 d_1 - m_2 d_0, \\ z_2^2 &= z_2\left(c_2 - m_3\right) - m_0 d_4 - m_1 d_3 - m_2 d_2 - m_3 d_1 - m_4 d_0, \\ &\vdots, \end{aligned} \tag{618}$$

which form a set of independent second order algebraic equations. When the z coefficients are applied into Eq. (34), all the even powers in the right hand side of the equation will cancel out with the even powers in the left hand side. Thus, the coefficients given in Eq. (618) to be solution of the Riccati sister equation, the coefficients of m, c and k must satisfy conditions given by the odd powers, which form a set of a equations.

A.2 FAMILIES OF RICCATI EQUATIONS WITH SIMPLE PARTICULAR SOLUTIONS USING INTEGRATION CONDITIONS

The given Riccati equation can be divided in 4 terms,

$$\underbrace{f_{2,1,2}^2}_{\phi_1} = \underbrace{m\dot{f}_{2,1,2}}_{\phi_2} + \underbrace{(c-\dot{m})\, f_{2,1,2}}_{\phi_3} - \underbrace{md}_{\phi_4}. \tag{619}$$

Therefore, one can solve this equation piecewise. *E.g.*,

- $\phi_1 = \phi_2$ and $\phi_3 = -\phi_4$

$$f_{2,1,2}^2 = m\dot{f}_{2,1,2} \implies f_{2,1,2} = \frac{-1}{a+\int \frac{1}{m}\,\mathrm{d}t}, \tag{620}$$

in which, a is a constant, and

$$(c-\dot{m})\, f_{2,1,2} = md \implies f_{2,1,2} = \frac{md}{c-\dot{m}}. \tag{621}$$

Comparing both equations, one finds a condition for d,

$$d = \frac{\dot{m}-c}{m\left(a+\int \frac{1}{m}\,\mathrm{d}t\right)}. \tag{622}$$

- $\phi_1 = \phi_4$ and $\phi_2 = -\phi_3$

$$m\dot{f}_{2,1,2} + (\dot{m}-c)\, f_{2,1,2} = 0 \implies f_{2,1,2} = \frac{a\exp\left(\int \frac{c}{m}\,\mathrm{d}t\right)}{m}, \tag{623}$$

and

$$f_{2,1,2}^2 = -md \implies f_{2,1,2} = \sqrt{-md}, \tag{624}$$

hence, the condition for d is

$$d = \frac{a \exp\left(2 \int \frac{c}{m} \mathrm{d}t\right)}{m^3}. \tag{625}$$

The negative sign and the square of constant a were omitted, since a can be any complex number.

- $\phi_1 = \phi_3$ and $\phi_2 = \phi_4$

$$f_{2,1,2}^2 = (c - \dot{m}) f_{2,1,2} \implies f_{2,1,2} = c - \dot{m}, \tag{626}$$

and,

$$m\dot{f}_{2,1,2} = md \implies d = \dot{f}_{2,1,2}, \tag{627}$$

thus,

$$d = \dot{c} - \ddot{m}. \tag{628}$$

Integration conditions, Eq. (622) and Eq. (625), can be more useful, since the constant a in them generate an infinite number of possible functions d.

A.3 CONVOLUTION OVER DIRAC'S DELTA DISTRIBUTION

The convolution of a function over the Dirac's delta is usually defined as (KANWAL, 2011)

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_k)\,dt = f(t_k). \tag{629}$$

The integration limits can be split as

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_k)\,dt = \int_{-\infty}^{0} f(t)\delta(t - t_k)\,dt + \int_{0}^{t} f(t)\delta(t - t_k)\,dt + \int_{t}^{\infty} f(t)\delta(t - t_k)\,dt, \tag{630}$$

for t_k strictly positive, the integral from $-\infty$ to 0 is 0 by definition, thus, the integral from 0 to t can be rewritten as

$$\int_0^t f(t)\delta(t-t_k)\,dt = \int_{-\infty}^{\infty} f(t)\delta(t-t_k)\,dt - \int_t^{\infty} f(t)\delta(t-t_k)\,dt. \tag{631}$$

The *filter* or *sifting* property of the delta of Dirac is due to the shape of this distribution, *i.e.*, it is null everywhere except in its discontinuity, thus, the function that multiplies the Dirac's delta is constant at this point, for the discontinuity of the delta distribution is infinitely close to the t_k point. Therefore, the value of the function can be taken out of the integral and the definition of the Dirac's delta is used to show that

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_k)\,dt = \int_{t_k-\tau}^{t_k+\tau} f(t)\delta(t-t_k)\,dt = f(t_k)\int_{t_k-\tau}^{t_k+\tau} \delta(t-t_k)\,dt = f(t_k), \tag{632}$$

hence, Equation (631) can be rewritten to

$$\int_0^t f(t)\delta(t-t_k)\,dt = f(t_k) - f(t_k)\begin{cases}0, & t \geq t_k \\ 1, & t < t_k\end{cases} = f(t_k)\mathscr{H}(t-t_k). \tag{633}$$

APPENDIX B – AUXILIARY FORMULATION FOR THE EXTENSION OF THE GENERALIZED INTEGRATING FACTOR FOR SYSTEMS OF LINEAR ODES WITH n DOFS

B.1 EVALUATION OF INTEGRATION CONSTANTS FOR COUPLED SYSTEMS OF ODES WITH CONSTANT COEFFICIENTS

The homogeneous solution with constant coefficients and when $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute is given by Eq. (223) if $\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}$ is non-singular. The complete solution

$$\mathbf{y}=\mathbf{y_h}+\mathbf{y_p}=\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2+\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t\right)\mathbf{C}_1+\mathbf{y_p}, \tag{634}$$

and its derivative w.r.t. time is given by

$$\dot{\mathbf{y}}=-\mathbf{F}_{2,1,1}\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2+\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t\right)\mathbf{C}_1+\dot{\mathbf{y}}_\mathbf{p}. \tag{635}$$

Considering initial conditions at a time t_0 yields

$$\mathbf{y}\left(t_0\right)=\exp\left(-\mathbf{F}_{2,1,1}t_0\right)\mathbf{C}_2+\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t_0\right)\mathbf{C}_1+\mathbf{y_p}\left(t_0\right), \tag{636}$$

and

$$\dot{\mathbf{y}}\left(t_0\right)=-\mathbf{F}_{2,1,1}\exp\left(-\mathbf{F}_{2,1,1}t_0\right)\mathbf{C}_2+\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t_0\right)\mathbf{C}_1+\dot{\mathbf{y}}_\mathbf{p}\left(t_0\right) \tag{637}$$

which can be summarized in a linear system

$$\begin{bmatrix}\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t_0\right) & \exp\left(-\mathbf{F}_{2,1,1}t_0\right)\\ \left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t_0\right) & -\mathbf{F}_{2,1,1}\exp\left(-\mathbf{F}_{2,1,1}t_0\right)\end{bmatrix}\begin{Bmatrix}\mathbf{C}_1\\ \mathbf{C}_2\end{Bmatrix}=\begin{Bmatrix}\mathbf{u}_0-\mathbf{y_p}\left(t_0\right)\\ \mathbf{v}_0-\dot{\mathbf{y}}_\mathbf{p}\left(t_0\right)\end{Bmatrix}. \tag{638}$$

Hence, $\mathbf{C}_1$ and $\mathbf{C}_2$ can be found by solving the above linear system with standard techniques, much alike the evaluation of C_1 and C_2 for a single degree of freedom problem.

The most common choice for t_0 is 0 such that Eq. (638) reduces to

$$\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ \left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right] & -\mathbf{F}_{2,1,1}\end{bmatrix}\begin{Bmatrix}\mathbf{C}_1\\ \mathbf{C}_2\end{Bmatrix}=\begin{Bmatrix}\mathbf{u}_0-\mathbf{y_p}\left(0\right)\\ \mathbf{v}_0-\dot{\mathbf{y}}_\mathbf{p}\left(0\right)\end{Bmatrix}. \tag{639}$$

Pivoting, it further simplifies to

$$\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{0} & \bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\end{bmatrix}\begin{Bmatrix}\mathbf{C}_1\\ \mathbf{C}_2\end{Bmatrix}=\begin{Bmatrix}\mathbf{u}_0-\mathbf{y_p}\left(0\right)\\ \mathbf{v}_0-\dot{\mathbf{y}}_\mathbf{p}\left(0\right)-\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]\left(\mathbf{u}_0-\mathbf{y_p}\left(0\right)\right)\end{Bmatrix}. \tag{640}$$

thus, it follows that

$$\mathbf{C}_2 = \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \left(\mathbf{v}_0 - \dot{\mathbf{y}}_{\mathbf{p}}(0) - \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right](\mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0))\right), \tag{641}$$

and

$$\mathbf{C}_1 = \mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0) - \mathbf{C}_2. \tag{642}$$

A very important particularization for previous equations is when evaluating the constants $\mathbf{C}_1$ and $\mathbf{C}_2$ for excitations described by Dirac's deltas, Heavisides and Heaviside series. As discussed in Chapter 2, $\mathbf{y}_p(t)$ and $\dot{\mathbf{y}}(t)_p$ are zero for $t \leq t_H$, where t_H is the first time with non null excitation. Therefore, if $t_0 \leq t_H$, both $\mathbf{y}_p(t_0)$ and $\dot{\mathbf{y}}_p(t_0)$ are $\mathbf{0}$ in Eq. (638), Eq. (641) and in Eq. (642) and there is no need to evaluate the derivative of the particular response with respect to time.

B.2 COMMUTATIVITY OF POWERS OF TWO MATRICES

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices $n \times n$ that commute. For positive integer powers a and b, the following property holds

$$\mathbf{A}^a\mathbf{B}^b = \mathbf{B}^b\mathbf{A}^a. \tag{643}$$

Proof.

$$\begin{aligned}
\mathbf{A}^a\mathbf{B}^b &= \underbrace{\mathbf{AAA}\ldots\mathbf{AAA}}_{a\ times}\underbrace{\mathbf{BBB}\ldots\mathbf{BBB}}_{b\ times} \\
&= \mathbf{AAA}\ldots\mathbf{AABABB}\ldots\mathbf{BBB} \\
&= \mathbf{AAA}\ldots\mathbf{ABABAB}\ldots\mathbf{BBB} \\
&= \mathbf{AAA}\ldots\mathbf{BABABA}\ldots\mathbf{BBB} \\
&= \mathbf{AAA}\ldots\mathbf{ABABAB}\ldots\mathbf{BBB} \\
&\vdots \\
\underbrace{\mathbf{BBB}\ldots\mathbf{BBB}}_{b\ times}\underbrace{\mathbf{AAA}\ldots\mathbf{AAA}}_{a\ times} &= \mathbf{B}^b\mathbf{A}^a.
\end{aligned}$$

□

B.3 COMMUTATIVITY OF INVERSE OF MATRIX

Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ square matrices that commute and $\mathbf{B}$ be invertible. Then

$$\mathbf{A}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{A}. \tag{644}$$

Proof.

$$\begin{aligned}
\mathbf{AB}^{-1} &= \mathbf{D},\\
\mathbf{A} &= \mathbf{DB}.\\
\mathbf{BA} = \mathbf{BDB} &= \mathbf{AB},\\
\mathbf{BD} &= \mathbf{A},\\
\mathbf{D} &= \mathbf{B}^{-1}\mathbf{A},\\
\implies \mathbf{AB}^{-1} &= \mathbf{B}^{-1}\mathbf{A}.
\end{aligned}$$

□

B.4 EXPONENTIAL MAP

An exponential map is defined as (GALLIER, 2011),

$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 + \ \dots\ + \frac{1}{n!}\mathbf{A}^n t^n = \sum_{j=0}^{\infty}\frac{1}{j!}\mathbf{A}^j t^j, \tag{645}$$

with time-derivative given by

$$\frac{d}{dt}\left[\exp(\mathbf{A}t)\right] = \mathbf{A} + \mathbf{A}^2 t + \ \dots\ + \frac{1}{(n-1)!}\mathbf{A}^n t^{n-1} = \sum_{j=0}^{\infty}\frac{1}{j!}\mathbf{A}^{j+1}t^j. \tag{646}$$

By direct comparison of Eq. (645) to Eq. (646), one immediately gets

$$\frac{d}{dt}\left[\exp(\mathbf{A}t)\right] = \mathbf{A}\exp(\mathbf{A}t) = \exp(\mathbf{A}t)\,\mathbf{A}. \tag{647}$$

B.5 EFFICIENT EVALUATION OF $\exp(\mathbf{A}t)$

One critical component of the proposed formulation is the efficient evaluation of $\exp(\mathbf{A}t)$ for various values of t. According to (APRAHAMIAN; HIGHAM, 2014), the following relation holds,

$$\exp(\mathbf{A})^{\alpha} = \exp(\alpha\mathbf{A}) \ \ \forall\, \alpha \in \mathbb{Z}, \tag{648}$$

for general matrix $\mathbf{A}$

Assuming that the time span $t \in [t_i, t_f]$ is discretized in n_t intervals Δt, it is possible to write $t = k\Delta t$ such that

$$\mathscr{E}(t) = \exp(\mathbf{A}t) = \exp(k\Delta t\mathbf{A}) = \exp(\Delta t\mathbf{A})^k, \tag{649}$$

where $k \in \mathbb{Z}$. Hence, the exponential map, $\mathscr{E}(t)$, in a discrete set of time can be evaluated through the following recursion

$$\begin{aligned} t_1: \quad & \mathscr{E}(t_1) = \exp(\Delta t \mathbf{A}) \\ t_2: \quad & \mathscr{E}(t_2) = \mathscr{E}(t_1)\exp(\Delta t \mathbf{A}) \\ t_3: \quad & \mathscr{E}(t_3) = \mathscr{E}(t_2)\exp(\Delta t \mathbf{A}) \\ & \vdots \\ t_k: \quad & \mathscr{E}(t_k) = \mathscr{E}(t_{k-1})\exp(\Delta t \mathbf{A}), \end{aligned} \tag{650}$$

where matrix $\exp(\Delta t \mathbf{A})$ has to evaluated just once.

B.6 COMMUTATIVITY OF EXPONENTIAL MAP AND MATRIX

Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ square matrices. If $\mathbf{A}$ and $\mathbf{B}$ commute, then

$$\exp(\mathbf{A})\mathbf{B} = \mathbf{B}\exp(\mathbf{A}). \tag{651}$$

Proof. By definition

$$\begin{aligned} \exp(\mathbf{A})\mathbf{B} &= \left[\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 + \ \dots \ + \frac{1}{n!}\mathbf{A}^nt^n\right]\mathbf{B} \\ &= \mathbf{IB} + \mathbf{AB}t + \frac{1}{2}\mathbf{A}^2\mathbf{B}t^2 + \ \dots \ + \frac{1}{n!}\mathbf{A}^n\mathbf{B}t^n \end{aligned} \tag{652}$$

such that

$$\exp(\mathbf{A})\mathbf{B} = \mathbf{BI} + \mathbf{BA}t + \frac{1}{2}\mathbf{BA}^2t^2 + \ \dots \ + \frac{1}{n!}\mathbf{BA}^nt^n = \mathbf{B}\exp(\mathbf{A}).$$

□

APPENDIX C – AUXILIARY FORMULATION FOR THE HEAVISIDE SERIES METHOD

C.1 CONDITIONS FOR THE SINGULARITY OF $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$

Using Eq. (239), the matrix $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is simplified to

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} = \bar{\mathbf{C}} - \bar{\mathbf{C}} - \sqrt{\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}} = -\sqrt{\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}}. \tag{653}$$

Using proportional damping and Eq. (680), one yields

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} = -\sqrt{\alpha^2\mathbf{I} + 2\alpha\beta\bar{\mathbf{K}} + \beta^2\bar{\mathbf{K}} - 4\bar{\mathbf{K}}} = -\sqrt{\alpha^2\mathbf{I} + (2\alpha\beta - 4)\,\bar{\mathbf{K}} + \beta^2\bar{\mathbf{K}}} \tag{654}$$

which, by recollecting Eq. (681), expands to

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} = -\sqrt{\alpha^2\mathbf{I} + (2\alpha\beta - 4)\,\boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} + \beta^2\boldsymbol{\Phi}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-1}}. \tag{655}$$

As $\boldsymbol{\Phi}\mathbf{I}\boldsymbol{\Phi}^{-1} = \boldsymbol{\Phi}\boldsymbol{\Phi}^{-1} = \mathbf{I}$ by definition, Equation (655) is simplified to

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} = -\sqrt{\boldsymbol{\Phi}\left(\alpha^2\mathbf{I} + (2\alpha\beta - 4)\,\boldsymbol{\Lambda} + \beta^2\boldsymbol{\Lambda}^2\right)\boldsymbol{\Phi}^{-1}} \tag{656}$$

according to (HIGHAM, 2008), $f\left(\mathbf{X}\mathbf{A}\mathbf{X}^{-1}\right) = \mathbf{X}f\left(\mathbf{A}\right)\mathbf{X}^{-1}$, thus,

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} = -\boldsymbol{\Phi}\sqrt{\alpha^2\mathbf{I} + (2\alpha\beta - 4)\,\boldsymbol{\Lambda} + \beta^2\boldsymbol{\Lambda}^2}\boldsymbol{\Phi}^{-1}. \tag{657}$$

As the matrix of eigenvalues, $\boldsymbol{\Lambda}$, is diagonal, Equation (657) can be further simplified using the property for diagonal matrix $f\left(diag\left(\mathbf{A}_k\right)\right) = diag\left(f\left(\mathbf{A}_k\right)\right)$, (HIGHAM, 2008),

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} = -\boldsymbol{\Phi} diag\left(\sqrt{\alpha^2 + (2\alpha\beta - 4)\,\lambda_k + \beta^2\lambda_k^2}\right)\boldsymbol{\Phi}^{-1}, \tag{658}$$

where λ_k is each one of the eigenvalues of $\bar{\mathbf{K}}$. One realizes that the eigenvalues of the matrix of the RHS of Eq. (658) are the values in the diagonal. As the LHS and RHS are similar matrices to each other, they share the same eigenvalues, hence, $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is singular if and only if the square root is null for some λ_k, *i.e.*,

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} \text{ singular} \iff \alpha^2 + (2\alpha\beta - 4)\,\lambda_k + \beta^2\lambda_k^2 = 0;\ k \subseteq \{1, 2, \ldots, n\}. \tag{659}$$

Since Equation (659) is a quadratic equation, it admits at most two roots, given by

$$\lambda_0 = \frac{2-\alpha\beta \pm 2\sqrt{1-\alpha\beta}}{\beta^2}. \tag{660}$$

As the eigenvalues are the natural frequencies squared in the context of FEM analysis, $\lambda_k = \omega_k^2$, it is not physical to have complex natural frequencies. A first and straightforward result is

$$\alpha\beta > 1 \implies \bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} \text{ non-singular.} \tag{661}$$

Nevertheless, one might conclude that, if proportional damping is used, at most two eigenvalues will be null if two natural frequencies match exactly with the square root of these two possible values of λ_0. Thus, another way of looking at Eq. (659) is substituting Eq. (692) into it,

$$\begin{aligned}\alpha^2 + (2\alpha\beta - 4)\lambda_k + \beta^2\lambda_k^2 = (\alpha + \beta\lambda_k)^2 - 4\lambda_k = \left(2\zeta_i\omega_i + \left(\lambda_k - \omega_i^2\right)\beta\right)^2 - 4\lambda_k \\ = \left(\lambda_k - \omega_i^2\right)^2\beta^2 + 4\zeta_i\omega_i\left(\lambda_k - \omega_i^2\right)\beta + 4\left(\zeta_i^2\omega_i^2 - \lambda_k\right),\end{aligned} \tag{662}$$

it immediately follows that, if $k = i$, Equation (662) will be zero if $\zeta_i = 1$, which characterizes critical damping. It is straightforward to prove that the same is true if $k = j$ and $\zeta_j = 1$, since α can be constructed in terms of the j-th natural frequency by $\alpha = 2\zeta_j\omega_j - \omega_j^2\beta$.

For nontrivial conditions to which Eq. (662) is null, one might find its roots as

$$\beta = 2\frac{\zeta_i\omega_i \pm \omega_k}{\omega_i^2 - \omega_k^2}, \tag{663}$$

that is almost equal to Eq. (691). Therefore, one can observe that, if $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is indeed singular, a minor change in ζ_i or ζ_j is enough to avoid the singularity and, consequently, the expensive usage of Jordan canonical form.

C.2 INITIAL CONDITIONS AND THE HEAVISIDE SERIES

Let a system of second order coupled ODEs have a function $\bar{\mathbf{f}}(t)$ multiplied by a Heaviside step as excitation function,

$$\mathbf{I}\ddot{\mathbf{y}}(t) + \bar{\mathbf{C}}\dot{\mathbf{y}}(t) + \bar{\mathbf{K}}\mathbf{y}(t) = \bar{\mathbf{f}}(t)\mathscr{H}(t - t_H), \tag{664}$$

which is the same as solving two systems of ODEs,

$$\begin{cases}\mathbf{I}\ddot{\mathbf{y}}_1(t) + \bar{\mathbf{C}}\dot{\mathbf{y}}_1(t) + \bar{\mathbf{K}}\mathbf{y}_1(t) = 0, & \text{if } t \le t_H \\ \mathbf{I}\ddot{\mathbf{y}}_2(t) + \bar{\mathbf{C}}\dot{\mathbf{y}}_2(t) + \bar{\mathbf{K}}\mathbf{y}_2(t) = \bar{f}(t). & \text{otherwise}\end{cases}, \tag{665}$$

It is straightforward that this holds true even when $t_H \to 0$. Therefore, all solutions given by Eq. (447), *i.e.* using HS as excitation function, have $\mathbf{y}_p(t) = \mathbf{0}$ and $\dot{\mathbf{y}}_p(t) = \mathbf{0}$ as fixed initial conditions. Thus, the imposition of non-homogeneous initial conditions, $\mathbf{y}(t_0) = \mathbf{u}_0$ and $\dot{\mathbf{y}}(t_0) = \mathbf{v}_0$, at $t_0 \leq t_H$ gets even simpler, through the following system of linear equations if $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ is non-singular (Appendix C.1),

$$\begin{bmatrix} \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t_0\right) & \exp\left(-\mathbf{F}_{2,1,1} t_0\right) \\ \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t_0\right) & -\mathbf{F}_{2,1,1} \exp\left(-\mathbf{F}_{2,1,1} t_0\right) \end{bmatrix} \begin{Bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \end{Bmatrix}. \tag{666}$$

which, particularized for $t_0 = 0$, simplifies to

$$\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] & -\mathbf{F}_{2,1,1} \end{bmatrix} \begin{Bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \end{Bmatrix}. \tag{667}$$

Pivoting, it further simplifies to

$$\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} \end{bmatrix} \begin{Bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 - \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] \mathbf{u}_0 \end{Bmatrix}. \tag{668}$$

whose solution is

$$\mathbf{C}_2 = \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \left(\mathbf{v}_0 - \dot{\mathbf{y}}_{\mathbf{p}}(0) - \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] \left(\mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0)\right)\right), \tag{669}$$

$$\mathbf{C}_1 = \mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0) - \mathbf{C}_2. \tag{670}$$

Thus, the constants $\mathbf{C}_1$ and $\mathbf{C}_2$ can be evaluated without any knowledge about the derivative of the particular response. If $\mathbf{F}^*_{2,1,1} = \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$, where * is the complex-conjugate operator, however, the relation between $\mathbf{C}_1$ and $\mathbf{C}_2$ is even simpler. To derive it, let $\mathbf{C}_2$ be substituted into $\mathbf{C}_1$ and the term $\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}$ be factorized,

$$\begin{aligned} \mathbf{C}_1 = \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \Big[\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] \left(\mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0)\right) - \mathbf{v}_0 + \dot{\mathbf{y}}_{\mathbf{p}}(0) \\ - \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] \left(\mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0)\right) \Big]. \end{aligned} \tag{671}$$

If $\mathbf{F}^*_{2,1,1} = \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$, then $\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] = \mathbf{F}^*_{2,1,1} - \mathbf{F}_{2,1,1}$ and, using the properties

$$\mathbf{C}_1 = \left[\mathbf{F}^*_{2,1,1} - \mathbf{F}_{2,1,1}\right]^{-1} \Big[\left[\mathbf{F}^*_{2,1,1} - \mathbf{F}_{2,1,1}\right] \left(\mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0)\right) - \mathbf{v}_0 + \dot{\mathbf{y}}_{\mathbf{p}}(0) - \mathbf{F}^*_{2,1,1} \left(\mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0)\right) \Big], \tag{672}$$

which simplifies to

$$\mathbf{C}_1 = \left[\mathbf{F}^*_{2,1,1} - \mathbf{F}_{2,1,1}\right]^{-1} \left[-\mathbf{v}_0 + \dot{\mathbf{y}}_{\mathbf{p}}(0) - \mathbf{F}_{2,1,1}\left(\mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0)\right)\right]. \tag{673}$$

Applying the complex-conjugate over Eq. (673), one gets

$$\mathbf{C}^*_1 = \left[\mathbf{F}^*_{2,1,1} - \mathbf{F}_{2,1,1}\right]^{-1} \left[\mathbf{v}_0 - \dot{\mathbf{y}}_{\mathbf{p}}(0) + \mathbf{F}^*_{2,1,1}\left(\mathbf{u}_0 - \mathbf{y}_{\mathbf{p}}(0)\right)\right] = \mathbf{C}_2, \tag{674}$$

hence, if $\mathbf{F}^*_{2,1,1} = \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$, one can calculate $\mathbf{C}_2$ using Eq. (669) and applying the conjugacy relation from Eq. (674) to evaluate $\mathbf{C}_1$. From the conjugacy relation between exponential maps explored in Chapter 3, the homogeneous response, Eq. (223), is

$$\mathbf{F}^*_{2,1,1} = \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] \implies \mathbf{y}_{\mathbf{h}}(t) = \exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2 + \exp\left(-\mathbf{F}_{2,1,1}t\right)^*\mathbf{C}_1, \tag{675}$$

which, by using Eq. (674), simplifies to

$$\begin{aligned}\mathbf{F}^*_{2,1,1} = \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] \implies \mathbf{y}_{\mathbf{h}}(t) &= \exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2 + \left(\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2\right)^* \\ &= 2\Re\left(\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2\right).\end{aligned} \tag{676}$$

C.2.1 Efficient computational evaluation of the homogeneous response

As discussed in Chapter 3 and in Appendix B.5, the exponential maps of the homogeneous responses can be evaluated in time points equally spaced from the point $t_0 = 0$, such that, for $t = t_i = i\Delta t$,

$$\exp\left(\mathbf{A}t_i\right)\mathbf{b} = \exp\left(\Delta t\mathbf{A}\right)^i\mathbf{b}. \tag{677}$$

Using this result, the evaluation of the homogeneous response can be summarized in Alg. 7.

Algorithm 7: Evaluation of the homogeneous response, $\mathbf{y_h}$, at equally spaced time points t_k

Calculate $\bar{\mathbf{K}}$, $\bar{\mathbf{C}}$, $\mathbf{F}_{2,1,1}$, $\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)$ and $\mathbf{C}_2$
if $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]=\mathbf{F}^*_{2,1,1}$ **then**
    $\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)=\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)^*$
    Evaluate the response at t_0
    $\mathbf{y_h}\left(t_0\right)=2\Re\left(\mathbf{C}_2\right)$
    for i=1,2,...,n_k
        Update the vector $\mathbf{C}_2$
        $\mathbf{C}_2=\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\mathbf{C}_2$
        Calculate the homogeneous response
        $\mathbf{y_h}\left(t_i\right)=2\Re\left(\mathbf{C}_2\right)$
    end
else
    Calculate $\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)$ and $\mathbf{C}_1$ using Eq. (669)
    Evaluate the homogeneous response at t_0
    $\mathbf{y_h}\left(t_0\right)=\mathbf{C}_1+\mathbf{C}_2$
    for i=1,2,...,n_k
        Update the vectors $\mathbf{C}_1$ and $\mathbf{C}_2$
        $\mathbf{C}_1=\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\Delta t\right)\mathbf{C}_1$
        $\mathbf{C}_2=\exp\left(-\mathbf{F}_{2,1,1}\Delta t\right)\mathbf{C}_2$
        Calculate the homogeneous response
        $\mathbf{y_h}\left(t_i\right)=\mathbf{C}_1+\mathbf{C}_2$
    end
end

C.3 EIGENVALUES OF $\mathbf{F}_{2,1,1}$ AND OF $\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}$

The objective of this section is to prove to which conditions the eigenvalues of $\mathbf{F}_{2,1,2}$ and of $\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}$ have positive real part. This is a necessary condition to the stability of the solution, since eigenvalues with negative real parts imply in negative damping.

If $\mathbf{C}$ and $\mathbf{K}$ commute,

$$\mathbf{F}_{2,1,1}=\frac{1}{2}\left(\bar{\mathbf{C}}+\sqrt{\bar{\mathbf{C}}^2-4\bar{\mathbf{K}}}\right), \tag{678}$$

hence,

$$\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}=\frac{1}{2}\left(\bar{\mathbf{C}}-\sqrt{\bar{\mathbf{C}}^2-4\bar{\mathbf{K}}}\right). \tag{679}$$

Assuming proportional damping, $\bar{\mathbf{C}}$ can be rewritten to

$$\bar{\mathbf{C}}=\mathbf{M}^{-1}\left(\alpha\mathbf{M}+\beta\mathbf{K}\right)=\alpha\mathbf{I}+\beta\mathbf{M}^{-1}\mathbf{K}=\alpha\mathbf{I}+\beta\bar{\mathbf{K}}. \tag{680}$$

Matrix $\bar{\mathbf{K}}$ has the same eigenvalues of the generalized eigenvalue problem defined by $(\mathbf{K}-\lambda\mathbf{M})\,\boldsymbol{\phi}=\mathbf{0}$, where $\lambda=\omega^2$. Thus, the eigenvalues λ are real and positive (ZIENKIEWICZ; TAYLOR; ZHU, 2013). Nevertheless, the matrix $\bar{\mathbf{K}}$ can be written in its diagonal form,

$$\bar{\mathbf{K}}=\boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}, \tag{681}$$

where $\boldsymbol{\Phi}$ is the matrix of eigenvectors and $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues.

Substituting Eq. (681) into Eq. (678), it yields

$$\mathbf{F}_{2,1,2}=\frac{1}{2}\left(\alpha\boldsymbol{\Phi}\boldsymbol{\Phi}^{-1}+\beta\boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}+\sqrt{\alpha^2\boldsymbol{\Phi}\boldsymbol{\Phi}^{-1}+(2\alpha\beta-4)\,\boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}+\beta^2\boldsymbol{\Phi}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-1}}\right), \tag{682}$$

which is further simplified to

$$\mathbf{F}_{2,1,2}=\frac{1}{2}\boldsymbol{\Phi}\left(\alpha\mathbf{I}+\beta\boldsymbol{\Lambda}+\sqrt{\alpha^2\mathbf{I}+(2\alpha\beta-4)\,\boldsymbol{\Lambda}+\beta^2\boldsymbol{\Lambda}^2}\right)\boldsymbol{\Phi}^{-1}. \tag{683}$$

From Eq. (683), one notices that $\mathbf{F}_{2,1,2}$ and the matrix between $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^{-1}$ are similar, thus, they share the same eigenvalues. As the eigenvalues λ in the diagonal of $\boldsymbol{\Lambda}$ are equal to ω^2, $\omega_1^2<\omega_2^2<\cdots<\omega_n^2$, Equation (683) can be simplified and the eigenvalues of $\mathbf{F}_{2,1,2}$ are

$$\gamma_k=\frac{1}{2}\left(\underbrace{\alpha+\beta\omega_k^2}_{R}+\sqrt{\underbrace{\alpha^2+(2\alpha\beta-4)\,\omega_k^2+\beta^2\omega_k^4}_{\Delta}}\right),\ k\in\{1,2,\ldots,n\}. \tag{684}$$

The same procedure can be done for the eigenvalues of $\bar{\mathbf{C}}-\mathbf{F}_{2,1,2}$, κ, with the only difference being the negative sign before the square root,

$$\kappa_k=\frac{1}{2}\left(\underbrace{\alpha+\beta\omega_k^2}_{R}-\sqrt{\underbrace{\alpha^2+(2\alpha\beta-4)\,\omega_k^2+\beta^2\omega_k^4}_{\Delta}}\right),\ k\in\{1,2,\ldots,n\}. \tag{685}$$

As Δ is always a real quantity, its square root can be either fully real either fully complex. Thus, by inspecting Eq. (684) and Eq. (685), the R term cannot be negative, otherwise the eigenvalues κ would not have positive real part. Nonetheless, if R is positive, again because of κ, the following condition must hold true,

$$\alpha^2+(2\alpha\beta-4)\,\omega_k^2+\beta^2\omega_k^4<\left(\alpha+\beta\omega_k^2\right)^2=\alpha^2+2\alpha\beta\omega_k^2+\beta^2\omega_k^4\implies-4\omega_k^2<0, \tag{686}$$

which is indeed true, since ω_k is a real quantity. Another interesting fact is that, from Eq. (686), for $R^2=\Delta$ and $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$ be a singular matrix, ω_k would have to be null, which is not physical, thus, $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$ is always non-singular for a physically acceptable system. Hence, for the eigenvalues to have positive real part,

$$\alpha+\beta\omega_k^2>0 \implies \Re(\gamma_k)>0, \Re(\kappa_k)>0; k\in\{1,2,\ldots,n\}. \tag{687}$$

Condition from Eq. (687) can be viewed as a polynomial function of ω_k,

$$p(\omega_k)=\beta\omega_k^2+\alpha, \tag{688}$$

that has one positive root and another negative root or both roots equal to zero. However, as in physical problems structural frequencies ω are always bigger than zero, just the positive root is of interest,

$$\omega_k=\sqrt{-\frac{\alpha}{\beta}}. \tag{689}$$

According to (RAO, 2017), the proportional coefficients α and β are determined by the linear system of equations defined by

$$\begin{aligned}\alpha+\omega_i^2\beta &= 2\zeta_i\omega_i\\ \alpha+\omega_j^2\beta &= 2\zeta_j\omega_j,\end{aligned} \tag{690}$$

where $\omega_1\leq\omega_j<\omega_i\leq\omega_n$. Solution is

$$\beta=2\frac{\zeta_i\omega_i-\zeta_j\omega_j}{\omega_i^2-\omega_j^2}, \tag{691}$$

$$\alpha=2\zeta_i\omega_i-\omega_i^2\beta=2\zeta_i\omega_i-2\omega_i^2\frac{\zeta_i\omega_i-\zeta_j\omega_j}{\omega_i^2-\omega_j^2}; \tag{692}$$

or for particular cases,

$$\begin{cases}\alpha=0 \implies \beta=\frac{2\zeta_i}{\omega_i},\\ \beta=0 \implies \alpha=2\zeta_i\omega_i;\end{cases} \tag{693}$$

it is trivially observed that, for this particular cases, Equation (687) holds true, thus,

$$\begin{cases}\alpha=0 \implies \beta=\frac{2\zeta_i}{\omega_i},\\ \beta=0 \implies \alpha=2\zeta_i\omega_i;\end{cases} \implies \Re(\gamma)>0, \Re(\kappa)>0. \tag{694}$$

The extreme point of $p(\omega_k)$ (either maximum or minimum) is unique and is located at 0, which is not a physical natural frequency. Thus, for the eigenvalues γ and κ to have real positive part, $p(\omega_k) > 0$ in the interval $[\omega_1, \omega_n]$. With some reasoning about a quadratic curve, one must realize that, if $\alpha \neq 0$ and $\beta \neq 0$, there are three possible cases,

1. The curve has upward concavity and cross the vertical axis above zero, *i.e.* $\alpha > 0$ and $\beta > 0$;
2. The curve has upward concavity and crosses the vertical axis below 0, thus, the curve must cross the horizontal axis before the first natural frequency, *i.e.* $\alpha < 0, \beta > 0 \implies \sqrt{\frac{-\alpha}{\beta}} < \omega_1$;
3. The curve has downward concavity and crosses the vertical axis above 0, thus the curve must cross the horizontal axis after the last natural frequency, *i.e.* $\alpha > 0, \beta < 0 \implies \sqrt{\frac{-\alpha}{\beta}} > \omega_n$;

summarizing and taking out the square root,

$$\begin{cases} \textbf{I)}\ \alpha > 0, \beta > 0, \\ \textbf{II)}\ \alpha < 0, \beta > 0 \implies -\frac{\alpha}{\beta} < \omega_1^2, \\ \textbf{III)}\ \alpha > 0, \beta < 0 \implies -\frac{\alpha}{\beta} > \omega_n^2. \end{cases} \tag{695}$$

The quotient of the damping coefficients can be expanded using Eq. (691) and (692),

$$-\frac{\alpha}{\beta} = \frac{\omega_i^2\beta - 2\zeta_i\omega_i}{\beta} = \omega_i^2 - \frac{2\zeta_i\omega_i}{\beta} = \omega_i^2 - \zeta_i\omega_i\frac{\omega_i^2 - \omega_j^2}{\zeta_i\omega_i - \zeta_j\omega_j}. \tag{696}$$

C.3.1 Case I, $\alpha > 0$ and $\beta > 0$

The term $\omega_i^2 - \omega_j^2$ is always positive since $\omega_i > \omega_j$, therefore, for β to be positive,

$$\zeta_i\omega_i > \zeta_j\omega_j. \tag{697}$$

For α, on the other hand,

$$\zeta_i\omega_i > \frac{\omega_i^2}{\omega_i^2 - \omega_j^2}\left(\zeta_i\omega_i - \zeta_j\omega_j\right), \tag{698}$$

rearranging the terms,

$$\zeta_i\omega_i\left(1 - \frac{\omega_i^2}{\omega_i^2 - \omega_j^2}\right) = \zeta_i\omega_i\left(\frac{\omega_i^2 - \omega_j^2 - \omega_i^2}{\omega_i^2 - \omega_j^2}\right) > -\frac{\omega_i^2}{\omega_i^2 - \omega_j^2}\zeta_j\omega_j. \tag{699}$$

As $\omega_i^2 - \omega_j^2$ is positive,

$$-\zeta_i \omega_i \omega_j^2 > -\zeta_j \omega_j \omega_i^2; \tag{700}$$

finally,

$$\zeta_i < \frac{\omega_i}{\omega_j} \zeta_j; \tag{701}$$

that, with Eq. (697), results in

$$\textbf{I)}\ \frac{\omega_j}{\omega_i} \zeta_j < \zeta_i < \frac{\omega_i}{\omega_j} \zeta_j. \tag{702}$$

Trivially, if $\zeta_i = \zeta_j$, condition **I** is true and the eigenvalues have all positive real parts.

C.3.2 Case II, $\alpha < 0$, $\beta > 0$

The condition for positive β was already stated in Eq. (697) and is simplified to

$$\zeta_i > \frac{\omega_j}{\omega_i} \zeta_j. \tag{703}$$

However, for negative α,

$$\zeta_i \omega_i < \frac{\omega_i^2}{\omega_i^2 - \omega_j^2} \left(\zeta_i \omega_i - \zeta_j \omega_j \right); \tag{704}$$

making the exact same steps from Eq. (699) to Eq. (701), the result is

$$\zeta_i > \frac{\omega_i}{\omega_j} \zeta_j. \tag{705}$$

Hence, if Equation (705) is true, so is Equation (703).

The second condition is

$$-\frac{\alpha}{\beta} = \omega_i^2 - \zeta_i \omega_i \frac{\omega_i^2 - \omega_j^2}{\zeta_i \omega_i - \zeta_j \omega_j} < \omega_1^2. \tag{706}$$

As β is positive,

$$\left(\omega_i^2 - \omega_1^2 \right) \left(\zeta_i \omega_i - \zeta_j \omega_j \right) < \zeta_i \omega_i \left(\omega_i^2 - \omega_j^2 \right); \tag{707}$$

separating by ζ_i,

$$\zeta_i\omega_i\left(\omega_1^2-\omega_j^2\right) > -\zeta_j\omega_j\left(\omega_i^2-\omega_1^2\right). \tag{708}$$

If $\omega_j = \omega_1$, both conditions in Eq. (705) and in Eq. (708) are true. Otherwise, $\omega_j > \omega_1 \implies \omega_1^2 - \omega_j^2 < 0$, thus,

$$\zeta_i\omega_i\left(\omega_j^2-\omega_1^2\right) < \zeta_j\omega_j\left(\omega_i^2-\omega_1^2\right); \tag{709}$$

and, finally,

$$\zeta_i < \zeta_j\frac{\omega_j\left(\omega_i^2-\omega_1^2\right)}{\omega_i\left(\omega_j^2-\omega_1^2\right)}. \tag{710}$$

For Equation (705) and Equation (710) to make sense together, the following statement must hold,

$$\frac{\omega_j\left(\omega_i^2-\omega_1^2\right)}{\omega_i\left(\omega_j^2-\omega_1^2\right)} > \frac{\omega_i}{\omega_j}, \tag{711}$$

which is rearranged to

$$\omega_j^2\left(\omega_i^2-\omega_1^2\right) > \omega_i^2\left(\omega_j^2-\omega_1^2\right), \tag{712}$$

thus,

$$\omega_j^2\omega_1^2 < \omega_i^2\omega_1^2, \tag{713}$$

that is true and, consequently,

$$\textbf{II)}\begin{cases}\omega_j = \omega_1 \implies \zeta_i > \frac{\omega_j}{\omega_i}\zeta_j,\\ \omega_j \neq \omega_1 \implies \zeta_j\frac{\omega_i}{\omega_j} < \zeta_i < \zeta_j\frac{\omega_j\left(\omega_i^2-\omega_1^2\right)}{\omega_i\left(\omega_j^2-\omega_1^2\right)}\end{cases}. \tag{714}$$

C.3.3 Case III, $\alpha > 0$, $\beta < 0$

It is straigthforward to see from Eq. (692) that, if β is negative, then, α is positive, so

$$\zeta_i < \frac{\omega_j}{\omega_i}\zeta_j \implies \alpha > 0. \tag{715}$$

Thereby, the remaining condition is

$$-\frac{\alpha}{\beta} = \omega_i^2 - \zeta_i \omega_i \frac{\omega_i^2 - \omega_j^2}{\zeta_i \omega_i - \zeta_j \omega_j} > \omega_n^2, \tag{716}$$

that is rearranged to

$$\omega_i^2 - \omega_n^2 > \zeta_i \omega_i \frac{\omega_i^2 - \omega_j^2}{\zeta_i \omega_i - \zeta_j \omega_j}, \tag{717}$$

and, once more, to

$$\omega_n^2 - \omega_i^2 < \zeta_i \omega_i \frac{\omega_i^2 - \omega_j^2}{\zeta_j \omega_j - \zeta_i \omega_i}. \tag{718}$$

Now, both sides of the inequality can be multiplied by the denominator,

$$\left(\omega_n^2 - \omega_i^2\right)\left(\zeta_j \omega_j - \zeta_i \omega_i\right) < \zeta_i \omega_i \left(\omega_i^2 - \omega_j^2\right); \tag{719}$$

again, trivially, if $\omega_n = \omega_i$, both conditions are true if only Eq. (715) is true. Otherwise, rearranging,

$$\zeta_i \omega_i \left(\omega_n^2 - \omega_j^2\right) > \left(\omega_n^2 - \omega_i^2\right) \zeta_j \omega_j, \tag{720}$$

and, finally,

$$\zeta_i > \frac{\omega_j \left(\omega_n^2 - \omega_i^2\right)}{\omega_i \left(\omega_n^2 - \omega_j^2\right)} \zeta_j. \tag{721}$$

As $\omega_i > \omega_j$,

$$\omega_n^2 - \omega_i^2 < \omega_n^2 - \omega_j^2 \implies \frac{\omega_n^2 - \omega_i^2}{\omega_n^2 - \omega_j^2} < 1, \tag{722}$$

so Equation (715) and Equation (721) work together and, henceforth,

$$\textbf{III)} \begin{cases} \omega_n = \omega_i \implies \zeta_i < \frac{\omega_j}{\omega_i} \zeta_j, \\ \omega_n \neq \omega_i \implies \zeta_j \frac{\omega_j\left(\omega_n^2 - \omega_i^2\right)}{\omega_i\left(\omega_n^2 - \omega_j^2\right)} < \zeta_i < \zeta_j \frac{\omega_j}{\omega_i} \end{cases}. \tag{723}$$

C.3.4 $\omega_j = \omega_1$ and $\omega_i = \omega_n$

In C.3.2 and in C.3.3, conditions regarding when $\omega_j = \omega_1$ and $\omega_i = \omega_n$ appeared separately and independently, but the question of what happens when both equalities are true simultaneously remains. To answer this question, let the solutions of α and β, Eq. (692) and Eq. (691), respectively, be rewritten in terms of ω_1 and ω_n,

$$\beta = 2\frac{\zeta_i\omega_n - \zeta_j\omega_1}{\omega_n^2 - \omega_1^2}, \tag{724}$$

$$\alpha = 2\omega_1\omega_n\frac{\zeta_j\omega_n - \zeta_i\omega_1}{\omega_n^2 - \omega_1^2}, \tag{725}$$

and, finally the quotient $-\frac{\alpha}{\beta}$, Eq. (696),

$$-\frac{\alpha}{\beta} = \omega_1\omega_n\frac{\zeta_i\omega_1 - \zeta_j\omega_n}{\zeta_i\omega_n - \zeta_j\omega_1}. \tag{726}$$

With these equations set, the three previous cases can be addresses separately again.

C.3.4.1 $\omega_j = \omega_1$ and $\omega_i = \omega_n$, case I

From Eq. (724), one observes that

$$\zeta_i\omega_n > \zeta_j\omega_1 \implies \zeta_i > \zeta_j\frac{\omega_1}{\omega_n} \implies \beta > 0, \tag{727}$$

and, from Eq. (725),

$$\zeta_i\omega_1 < \zeta_j\omega_n \implies \zeta_i < \zeta_j\frac{\omega_n}{\omega_1} \implies \alpha > 0, \tag{728}$$

consequently,

$$\zeta_j\frac{\omega_1}{\omega_n} < \zeta_i < \zeta_j\frac{\omega_n}{\omega_1} \implies \alpha,\beta > 0. \tag{729}$$

C.3.4.2 $\omega_j = \omega_1$ and $\omega_i = \omega_n$, case II

From Eq. (725), follows that

$$\zeta_i\omega_1 > \zeta_j\omega_n \implies \zeta_i > \zeta_j\frac{\omega_n}{\omega_1} \implies \alpha < 0, \tag{730}$$

and adding to Eq. (727),

$$\zeta_i > \zeta_j \frac{\omega_n}{\omega_1} \implies \alpha < 0, \beta > 0. \tag{731}$$

Going back to the quotient condition in Eq. (695) and using Eq. (726), one observes that

$$-\frac{\alpha}{\beta} = \omega_1 \omega_n \frac{\zeta_i \omega_1 - \zeta_j \omega_n}{\zeta_i \omega_n - \zeta_j \omega_1} < \omega_1^2. \tag{732}$$

The inequality can be multiplied in both sides by $\frac{\zeta_i \omega_n - \zeta_j \omega_1}{\omega_1}$ without any prejudice of sign, since β is positive, which means that the $\zeta_i \omega_n - \zeta_j \omega_1 > 0$,

$$\omega_n \left(\zeta_i \omega_1 - \zeta_j \omega_n\right) < \omega_1 \left(\zeta_i \omega_n - \zeta_j \omega_1\right), \tag{733}$$

which can be simplified to

$$-\zeta_j \omega_n^2 < -\zeta_j \omega_1^2, \tag{734}$$

and, by multiplying both sides by -1,

$$\zeta_j \omega_n^2 > \zeta_j \omega_1^2, \tag{735}$$

that is true since $\omega_n > \omega_1$.

C.3.4.3 $\omega_j = \omega_1$ and $\omega_i = \omega_n$, case III

From Eq. (724), follows that

$$\zeta_i \omega_n < \zeta_j \omega_1 \implies \zeta_i < \zeta_j \frac{\omega_1}{\omega_n} \implies \beta < 0, \tag{736}$$

and adding to Eq. (728),

$$\zeta_i < \zeta_j \frac{\omega_1}{\omega_n} \implies \alpha > 0, \beta < 0. \tag{737}$$

Going back to the quotient condition in Eq. (695) and using Eq. (726), one observes that

$$-\frac{\alpha}{\beta} = \omega_1 \omega_n \frac{\zeta_i \omega_1 - \zeta_j \omega_n}{\zeta_i \omega_n - \zeta_j \omega_1} > \omega_n^2. \tag{738}$$

The inequality can be multiplied in both sides by $\frac{\zeta_i\omega_n-\zeta_j\omega_1}{\omega_n}$ and have its comparison operator flipped, since β is negative, which means that the $\zeta_i\omega_n-\zeta_j\omega_1<0$,

$$\omega_1\left(\zeta_i\omega_1-\zeta_j\omega_n\right)<\omega_n\left(\zeta_i\omega_n-\zeta_j\omega_1\right), \tag{739}$$

which can be simplified to

$$\zeta_j\omega_1^2<\zeta_j\omega_n^2, \tag{740}$$

which is true since $\omega_1<\omega_n$.

C.3.5 Final remarks

One can verify in Eq. (691) and in Eq. (692), respectively, that

$$\zeta_i=\zeta_j\frac{\omega_j}{\omega_i}\implies\beta=0\implies\alpha>0, \tag{741}$$

$$\zeta_i=\zeta_j\frac{\omega_i}{\omega_j}\implies\alpha=0\implies\beta>0, \tag{742}$$

which, according to Eq. (694), implies that

$$\begin{cases}\zeta_i=\zeta_j\frac{\omega_j}{\omega_i}\implies\beta=0\implies\alpha>0\\ \zeta_i=\zeta_j\frac{\omega_i}{\omega_j}\implies\alpha=0\implies\beta>0\end{cases}\implies\Re(\gamma),\Re(\kappa)>0. \tag{743}$$

Adding this information to the results found in Eq. (702), Eq. (714) and in Eq. (723), it yields that

$$\begin{cases}i.\ \omega_i\neq\omega_n,\omega_j\neq\omega_1\implies\zeta_j\frac{\omega_j\left(\omega_n^2-\omega_i^2\right)}{\omega_i\left(\omega_n^2-\omega_j^2\right)}<\zeta_i<\zeta_j\frac{\omega_j\left(\omega_i^2-\omega_1^2\right)}{\omega_i\left(\omega_j^2-\omega_1^2\right)},\\ ii.\ \omega_i=\omega_n,\omega_j\neq\omega_1\implies\zeta_i<\frac{\omega_j}{\omega_i}\zeta_j,\\ iii.\ \omega_j=\omega_1,\omega_i\neq\omega_n\implies\zeta_i>\frac{\omega_j}{\omega_i}\zeta_j,\\ iv.\ \omega_j=\omega_1,\omega_i=\omega_n\implies\Re(\gamma),\Re(\kappa)>0\end{cases}\implies\Re(\gamma),\Re(\kappa)>0. \tag{744}$$

To prove that the first interval is consistent,

$$\frac{\left(\omega_i^2-\omega_1^2\right)}{\left(\omega_j^2-\omega_1^2\right)}>\frac{\left(\omega_n^2-\omega_i^2\right)}{\left(\omega_n^2-\omega_j^2\right)}; \tag{745}$$

as both denominators are positive by construction,

$$\left(\omega_i^2-\omega_1^2\right)\left(\omega_n^2-\omega_j^2\right)>\left(\omega_n^2-\omega_i^2\right)\left(\omega_j^2-\omega_1^2\right), \tag{746}$$

and expanding both sides,

$$\omega_i^2\omega_n^2-\omega_i^2\omega_j^2-\omega_1^2\omega_n^2+\omega_1^2\omega_j^2>\omega_n^2\omega_j^2-\omega_n^2\omega_1^2-\omega_i^2\omega_j^2+\omega_i^2\omega_1^2. \tag{747}$$

Which, by rearranging the terms, becomes

$$\omega_n^2\left(\omega_i^2-\omega_j^2\right)>\omega_1^2\left(\omega_i^2-\omega_j^2\right), \tag{748}$$

that is a true statement.

APPENDIX D – AUXILIARY FORMULATION FOR THE EXTENSION OF THE GIF AND THE HS METHODS FOR NON-CLASSICAL NORMAL MODES

D.1 SYLVESTER EQUATION WHEN THE MODES ARE CLASSICAL NORMAL

Let Equation (523) be rearranged and the definition for $\mathbf{X}$ from Eq. (522) be substituted into it,

$$\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\mathbf{X}-\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t \\ \exp\left(\mathbf{F}_{2,1,1}t\right)\mathbf{F}_{2,1,1}=\mathbf{I}. \tag{749}$$

It was shown in Chapter 3 that if the system of equations given by Eq. (1) has classical normal modes, the quadratic matrix equation from Eq. (514) has a solution of the form of Eq. (515) and that $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}}$ commute. It was also proven in Appendix B.6 that if two matrices commute, so does one of these matrices with the exponential of the other. For this reason, it is clear that

$$\mathbf{F}_{2,1,1}\bar{\mathbf{C}}=\bar{\mathbf{C}}\mathbf{F}_{2,1,1}\implies\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\mathbf{X}-\mathbf{F}_{2,1,1}\exp\left(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right) \\ \int\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathrm{d}t\exp\left(\mathbf{F}_{2,1,1}t\right)=\mathbf{I}. \tag{750}$$

Henceforth,

$$\mathbf{F}_{2,1,1}\bar{\mathbf{C}}=\bar{\mathbf{C}}\mathbf{F}_{2,1,1}\implies\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\mathbf{X}-\mathbf{F}_{2,1,1}\mathbf{X}=\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]\mathbf{X}=\mathbf{I}, \tag{751}$$

which implies that

$$\mathbf{F}_{2,1,1}\bar{\mathbf{C}}=\bar{\mathbf{C}}\mathbf{F}_{2,1,1}\implies\mathbf{X}=\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}. \tag{752}$$

Equation (752) means that the original Sylvester equation, Eq. (523), has the above unique solution when the the system presents classical normal modes. However, for this solution to exist, the inverse of $\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]$ must also exist, and this matrix has to be non-singular. Conditions for this matrix to be singular were already discussed in Chapter 4 and it was observed that these conditions relate to the system being critically damped. As it was said prior, for a Sylvester equation to have a unique solution, the coefficients cannot share eigenvalues (BARTELS; STEWART, 1972), thus, it is hypothesized that, when the system does not have classical normal modes and it is critically damped, $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$ and $\mathbf{F}_{2,1,1}$ will share one or more eigenvalues. The motivation for this hypothesis is in Eq. (752), since the matrix $\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]$ is singular when $\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]$ and $\mathbf{F}_{2,1,1}$ share eigenvalues.